

Intrinsic Affine Positionality in Ordered Regular Languages: Fixed-Width and Genealogical Rank

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Abstract

Let L be an infinite prefix-closed, right-prolongable regular language over a finite ordered alphabet of nonnegative integer digits. We ask when the lexicographic ranks of its fixed-width cross-sections, or more generally unit-spaced graded ranks, admit a normalized scalar positional representation

$$V_U(a_1 \cdots a_n) = \sum_{j=1}^n a_j U_{n-j}, \quad U_0 = 1.$$

Positionality at one root propagates to every live residual state. There are residual translations $\lambda_q(n)$ such that

$$R_q^{(n)}(w) = V_U(w) - \lambda_q(n),$$

and this residual affine form is equivalent to local tiling equations. If a_q is the least legal digit, $p(q) = \delta(q, a_q)$, and adjacent digits have targets r, s , then

$$\lambda_q(n+1) = a_q U_n + \lambda_{p(q)}(n), \quad U_n = N_r(n) + \lambda_r(n) - \lambda_s(n).$$

With at least one branching residual state, any genuine adjacency determines the only possible normalized basis; the basis and all residual translations are then unique and integral whenever compatibility holds.

The translation subsystem has a canonical finite quotient. Two states are equivalent when their infinite least-digit threshold itineraries agree. If the minimal live residual DFA has s states and this lower-itinerary quotient has b classes, census dynamics require s coordinates while translations require only b . The resulting $(s+b)$ -dimensional realization has one common-translation gauge mode invisible to positional value. Hence

$$r_U \leq s + b - 1, \quad r_{\beta^{(q)}} \leq s + b,$$

where r denotes Hankel rank and $\beta^{(q)} = \lambda_q$ is the residual grading boundary. If compatibility fails, a failure appears by continuation depth at most $s+b-2$. This recognition bound is not universally sharp: for genuine integer-digit systems the exact radius at $(s, b) = (4, 4)$ is 5, whereas the rational-threshold relaxation attains depth 6. Thus finite recognition has an arithmetic component not visible in the ambient affine realization dimension.

Both complexity bounds are sharp for every $1 \leq b \leq s$:

$$R_U(s, b) = s + b - 1, \quad R_{\beta}(s, b) = s + b.$$

Thus maximal positional recurrence complexity splits into census complexity plus lower-itinerary complexity, with one dimension removed by the gauge.

Fixed-width and genealogical positionality are root-specific specializations of the same intrinsic structure. Fixed-width rank occurs exactly when the chosen root boundary is zero. Genealogical rank occurs exactly when it equals the cumulative census $B_n = \sum_{m < n} |L \cap \Sigma^m|$, in which case the least root digit is 1 and

$$U(z) = \frac{C(z)}{(1-z)g_*(z)}.$$

For positive bases, both canonical root conventions have Perron-scale growth, while general intrinsic boundaries may support strictly faster positive growth.

1. Introduction

A positional numeration system evaluates a digit word

$$a_1 \cdots a_n$$

through a place-value sequence $U = (U_n)_{n \geq 0}$:

$$V_U(a_1 \cdots a_n) = \sum_{j=1}^n a_j U_{n-j}.$$

Usually U is specified first and one then studies the language of legal representations. Here the direction is reversed. We begin with an ordered regular language and ask whether its rank structure determines such a positional valuation.

Let $L \subseteq \Sigma^*$ be an infinite regular language over a finite ordered alphabet of nonnegative integer digits. For each width n , the cross-section

$$L(n) = L \cap \Sigma^n$$

has its zero-based lexicographic rank $R^{(n)}$. The fixed-width problem asks whether one normalized sequence U , independent of n , satisfies

$$R^{(n)}(w) = V_U(w)$$

for every legal word w of width n .

Genealogical rank differs only by an offset. Words are ordered first by length and then lexicographically, so a word $w \in L(n)$ has rank

$$G(w) = \sum_{m < n} |L(m)| + R^{(n)}(w).$$

More generally, any unit-spaced scoring rule that preserves lexicographic order within each width has the form

$$G_n(w) = \beta_n + R^{(n)}(w)$$

for a unique boundary sequence β . Fixed-width rank corresponds to $\beta = 0$, while genealogical rank corresponds to the cumulative census

$$B_n = \sum_{m < n} |L(m)|.$$

The central point of this paper is that these are not separate positionality theories. They are two root conditions on one residual affine structure.

We work with prefix-closed, right-prolongable languages. Hence every live residual admits continuations of every length. This synchronizes all residual states to one width parameter and allows a single sequence U to act throughout the automaton.

Suppose positionality holds at one root. It then propagates to every live residual: for each residual state q there is a translation $\lambda_q(n)$ such that

$$R_q^{(n)}(w) = V_U(w) - \lambda_q(n).$$

Width one forces each legal digit set to be an interval

$$D(q) = \{a_q, a_q + 1, \dots, b_q\}.$$

Writing $p(q) = \delta(q, a_q)$, the residual translations satisfy

$$\lambda_q(n+1) = a_q U_n + \lambda_{p(q)}(n),$$

and adjacent digit cylinders with targets r, s satisfy

$$U_n = N_r(n) + \lambda_r(n) - \lambda_s(n).$$

These local equations are also sufficient. They therefore give a root-free characterization of affine positionality.

Branching makes the structure rigid. Any genuine adjacency recursively determines the normalized basis from $U_0 = 1$. If the remaining adjacencies are compatible, the basis and all residual translations are unique and integral.

The translations themselves have a smaller state space than the full residual DFA. They depend only on the lower-threshold itinerary

$$\iota(q) = (a_q, a_{p(q)}, a_{p^2(q)}, \dots).$$

Let s be the number of states in the minimal live residual DFA and let b be the number of distinct lower-itinerary classes. Census dynamics remain s -dimensional, but the translation subsystem compresses to b coordinates. The resulting affine realization has dimension $s + b$.

One direction in this realization is a common translation mode. It is invisible to positional value but visible to residual boundaries. This gives the central bounds

$$r_U \leq s + b - 1, \quad r_{\beta(q)} \leq s + b,$$

where r denotes Hankel rank. Both are sharp for every admissible pair $1 \leq b \leq s$:

$$R_U(s, b) = s + b - 1, \quad R_\beta(s, b) = s + b.$$

Thus the maximal recurrence complexity separates into census complexity and lower-itinerary complexity, with one dimension lost from positional value through the common-translation gauge.

The same finite realization gives a recognition theorem. If a candidate basis fails an adjacency compatibility condition, some failure appears by continuation depth at most

$$s + b - 2.$$

This dimension bound need not be attained by genuine digit automata. Writing $H_{\mathbb{Z}}(s, b)$ for the exact integer-digit recognition radius, we prove

$$\boxed{H_{\mathbb{Z}}(4, 4) = 5 < 6.}$$

In the rational-threshold relaxation the missing step reappears:

$$\boxed{H_{\mathbb{Q}}(4, 4) = 6.}$$

The defect is therefore arithmetic rather than a reduction of the ambient affine realization dimension. The lower-itinerary parameter itself is finite and computable: equality of the first s itinerary symbols already decides equality of the infinite itineraries.

Fixed-width and genealogical positionality are then short corollaries of the intrinsic theory. Fixed-width rank occurs exactly when the chosen root boundary vanishes. Genealogical rank occurs exactly when the root boundary equals B ; this forces the least root digit to be 1 and yields

$$U(z) = \frac{C(z)}{(1-z)g_*(z)}.$$

For positive bases, both canonical root conventions have Perron-scale growth. General intrinsic boundaries need not: we give a positive example whose place values grow strictly faster than the language census.

1.1 Main results

The proof is organized around seven results.

Affine Rank Characterization. Root positionality propagates to every live residual and is equivalent to interval digit sets, the lower recursion, and adjacent-cylinder tiling.

Intrinsic Rigidity. With branching, one adjacency determines the only possible normalized basis; the basis and all residual translations are integral whenever compatibility holds.

Lower-Itinerary Quotient. Two states have the same translation exactly when their lower-threshold itineraries agree. The quotient has b classes and is computable by finite partition refinement.

Compressed Realization and Recognition. The coupled census-translation system has dimension $s + b$. Positional value factors through a quotient of dimension $s + b - 1$, and incompatibility is detected by depth $s + b - 2$. The bound is not universally sharp for integer digits: $H_{\mathbb{Z}}(4, 4) = 5$, while the rational relaxation has $H_{\mathbb{Q}}(4, 4) = 6$.

Sharp Two-Parameter Complexity. For every $1 \leq b \leq s$,

$$R_U(s, b) = s + b - 1, \quad R_{\beta}(s, b) = s + b.$$

Twisted Coboundary and Residual Transfer. Edge defects form an exact twisted coboundary, yielding path and root-change formulas.

Canonical Root Boundaries. Fixed-width and genealogical positionality are respectively the zero-boundary and cumulative-census specializations. Positive bases in both cases have Perron-scale growth.

1.2 Relation to earlier numeration theory

Abstract numeration systems enumerate an infinite regular language genealogically and use that order to represent the nonnegative integers. Lecomte and Rigo developed the regular-language framework and its recognizability theory [1,2]. Recent work on Dumont-Thomas systems addresses positionality within a substantial substitution-generated subclass [4].

The present problem is complementary. We begin with fixed-width lexicographic ranks of residual languages, derive the local positional structure, and only afterwards impose a global ranking convention through the root boundary. This separates within-width rank geometry from the offsets contributed by other widths.

The parameter b is specific to this residual-affine viewpoint. Ordinary state count records distinct residual languages; b records distinct lower-threshold translation behaviours. The coupled (s, b) structure, rather than census rationality alone, controls positional recurrence complexity.

The rational-series tools used below are standard. What is new here is the automaton-specific compression that precedes ordinary reachability and observability minimization, together with the resulting sharp two-parameter bounds.

2. Ordered residual automata and graded scores

2.1 Standing hypotheses

Let $\Sigma \subseteq \mathbb{Z}_{\geq 0}$ be a finite nonempty alphabet equipped with its natural order. Let

$$L \subseteq \Sigma^*$$

be an infinite regular language satisfying:

1. **prefix closure:** every prefix of a word in L lies in L ;
2. **right prolongability:** every word in L has at least one one-symbol extension in L .

Let $\mathcal{A} = (Q, q_*, \delta)$ be the accessible minimal partial DFA of live residual states, with the rejecting sink deleted. Thus every state $q \in Q$ represents a distinct live residual language L_q . Write

$$s = |Q|.$$

For a state q , let

$$D(q) = \{a \in \Sigma : \delta(q, a) \text{ is live}\}$$

be its legal digit set. Right prolongability implies $D(q) \neq \emptyset$ for every q .

For $n \geq 0$, define the width- n residual language

$$L_q(n) = L_q \cap \Sigma^n$$

and its census

$$N_q(n) = |L_q(n)|.$$

Since every live residual is right-prolongable,

$$N_q(n) \geq 1 \quad (q \in Q, n \geq 0).$$

In particular,

$$N_q(0) = 1.$$

Order each $L_q(n)$ lexicographically and let

$$R_q^{(n)} : L_q(n) \rightarrow \{0, 1, \dots, N_q(n) - 1\}$$

be its zero-based rank function.

Let A denote the residual census matrix,

$$A_{qr} = \#\{a \in D(q) : \delta(q, a) = r\}.$$

Writing $\mathbf{N}_n = (N_q(n))_{q \in Q}$,

$$\mathbf{N}_0 = \mathbf{1}, \quad \mathbf{N}_{n+1} = A\mathbf{N}_n.$$

2.2 Normalized positional evaluation

Let

$$U = (U_n)_{n \geq 0}$$

be a scalar sequence normalized by

$$\boxed{U_0 = 1.}$$

For a digit word $w = a_1 \cdots a_n$, define

$$V_U(w) = \sum_{j=1}^n a_j U_{n-j}, \quad V_U(\varepsilon) = 0.$$

The normalization $U_0 = 1$ is structural. Without it, width-one positionality would force legal digits into an arithmetic progression whose step is $1/U_0$. With this normalization, width-one rank will force consecutive integer digits.

2.3 Graded lex-compatible scoring

For each width n , let

$$G_n : L(n) \rightarrow \mathbb{Z}$$

be a score. We call $G = (G_n)$ **unit-spaced lex-compatible** if consecutive words in the lexicographic order on $L(n)$ receive consecutive integer scores.

Proposition 2.1. Ordering Reduction A score family G is unit-spaced lex-compatible if and only if there is a unique integer sequence $\beta = (\beta_n)$ such that

$$\boxed{G_n(w) = \beta_n + R_{q_*}^{(n)}(w)} \tag{2.1}$$

for every n and every $w \in L(n)$.

Proof Let $w_n^{\min}$ be the lexicographically first word of $L(n)$, and put

$$\beta_n = G_n(w_n^{\min}).$$

Unit spacing forces the score to increase by exactly one at each lexicographic successor, giving (2.1). Uniqueness follows by evaluating at $w_n^{\min}$. The converse is immediate. $\square$

A unit-spaced graded score is **normalized scalar positional** if there is a sequence U , $U_0 = 1$, such that

$$\boxed{G_n(w) = V_U(w)} \tag{2.2}$$

for every legal root word. Combining (2.1) and (2.2),

$$\boxed{R_{q_*}^{(n)}(w) = V_U(w) - \beta_n.} \tag{2.3}$$

At width zero this forces $\beta_0 = 0$.

Two scoring conventions will be central later. Fixed-width rank has

$$\beta_n = 0.$$

Genealogical rank has

$$\beta_n = B_n := \sum_{m=0}^{n-1} N_{q_*}(m).$$

The latter is the unique global unit-spaced ranking that orders all shorter words before all longer words and uses lexicographic order inside each width.

3. Propagation and local affine characterization

The main structural theorem begins from the root identity (2.3) and removes the root from the problem.

Theorem 3.1. Affine Rank Characterization

Assume the standing hypotheses and $U_0 = 1$.

The following are equivalent.

(i) Root positionality. There is a sequence β , $\beta_0 = 0$, such that

$$R_{q_*}^{(n)}(w) = V_U(w) - \beta_n$$

for every root word $w \in L(n)$.

(ii) Residual affine positionality. For every live residual state q there is a sequence λ_q , with $\lambda_q(0) = 0$, such that

$$\boxed{R_q^{(n)}(w) = V_U(w) - \lambda_q(n)} \tag{3.1}$$

for all n and all $w \in L_q(n)$. At the root,

$$\lambda_{q_*} = \beta.$$

(iii) Local affine tiling. Every legal digit set is an interval

$$D(q) = \{a_q, a_q + 1, \dots, b_q\}. \tag{3.2}$$

Writing

$$p(q) = \delta(q, a_q),$$

there are sequences λ_q , $\lambda_q(0) = 0$, satisfying

$$\boxed{\lambda_q(n+1) = a_q U_n + \lambda_{p(q)}(n)} \tag{3.3}$$

for every state and width, and whenever $d, d+1 \in D(q)$ have targets

$$r = \delta(q, d), \quad s = \delta(q, d+1),$$

one has

$$\boxed{U_n = N_r(n) + \lambda_r(n) - \lambda_s(n)}. \quad (3.4)$$

Moreover, whenever these conditions hold,

$$\boxed{V_U(L_q(n)) = \lambda_q(n) + \{0, 1, \dots, N_q(n) - 1\}}. \quad (3.5)$$

Proof We prove $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (ii)$; the implication $(ii) \Rightarrow (i)$ is obtained by restricting to the root.

Root-to-residual propagation. Fix a state q and an access word $x = x_1 \cdots x_k$ taking q_* to q . For continuation width n , let $S_x(n)$ be the number of root words of width $k+n$ lying lexicographically before the cylinder $xL_q(n)$. Then

$$R_{q_*}^{(k+n)}(xw) = S_x(n) + R_q^{(n)}(w)$$

for $w \in L_q(n)$. Put

$$C_x(n) = \sum_{i=1}^k x_i U_{k+n-i}.$$

Since

$$V_U(xw) = C_x(n) + V_U(w),$$

root positionality gives

$$R_q^{(n)}(w) = V_U(w) - \lambda_{x,q}(n),$$

where

$$\lambda_{x,q}(n) = \beta_{k+n} + S_x(n) - C_x(n). \quad (3.6)$$

If y is another access word to q , right prolongability supplies at least one $w \in L_q(n)$ for every n . Evaluating the two expressions for the same residual rank shows

$$\lambda_{x,q}(n) = \lambda_{y,q}(n).$$

Thus λ_q is well defined. At $n=0$, root positionality applied to the access word itself yields $\lambda_q(0) = 0$. This proves (ii), with $\lambda_{q_*} = \beta$.

Interval legality. Assume (ii). Write the legal one-digit set at q in increasing order as

$$D(q) = \{d_0 < d_1 < \cdots < d_{k-1}\}.$$

Since

$$R_q^{(1)}(d_i) = i, \quad V_U(d_i) = d_i U_0 = d_i,$$

(3.1) gives

$$i = d_i - \lambda_q(1).$$

Hence

$$d_i = \lambda_q(1) + i,$$

so $D(q)$ is an interval of consecutive integers and

$$a_q = \lambda_q(1).$$

Lower recursion. Let $w \in L_{p(q)}(n)$. The word $a_q w$ lies in the first digit cylinder, so

$$R_q^{(n+1)}(a_q w) = R_{p(q)}^{(n)}(w).$$

Using (3.1) on both sides and cancelling $V_U(w)$ gives (3.3).

Adjacent tiling. Let adjacent digits $d, d+1$ have targets r, s . The first width- n continuation in each target residual has rank zero and therefore positional values $\lambda_r(n)$ and $\lambda_s(n)$. The first word of the $(d+1)$ -cylinder occurs exactly $N_r(n)$ ranks after the first word of the d -cylinder. Hence

$$N_r(n) = (d+1)U_n + \lambda_s(n) - dU_n - \lambda_r(n),$$

which is (3.4). Thus (iii) holds.

Converse. Assume (iii). We prove (3.1) by induction on n . The case $n = 0$ is immediate. Assume the result at width n , and take

$$dw \in L_q(n+1), \quad w \in L_{q_d}(n),$$

where $q_d = \delta(q, d)$. Lexicographic rank decomposes as

$$R_q^{(n+1)}(dw) = \sum_{c=a_q}^{d-1} N_{q_c}(n) + R_{q_d}^{(n)}(w). \quad (3.7)$$

By induction,

$$R_{q_d}^{(n)}(w) = V_U(w) - \lambda_{q_d}(n).$$

Define

$$F_d = \sum_{c=a_q}^{d-1} N_{q_c}(n) - \lambda_{q_d}(n).$$

At the least digit,

$$F_{a_q} = -\lambda_{p(q)}(n) = a_q U_n - \lambda_q(n+1)$$

by (3.3). For adjacent legal digits,

$$F_{d+1} - F_d = N_{q_d}(n) + \lambda_{q_d}(n) - \lambda_{q_{d+1}}(n) = U_n$$

by (3.4). Hence

$$F_d = dU_n - \lambda_q(n+1).$$

Substituting into (3.7),

$$R_q^{(n+1)}(dw) = dU_n + V_U(w) - \lambda_q(n+1) = V_U(dw) - \lambda_q(n+1).$$

This completes the induction and proves (ii).

Finally, (3.5) follows immediately from (3.1) because residual ranks run through $0, \dots, N_q(n) - 1$. $\square$

Remark 3.2. Root independence

The local conditions (3.2)–(3.4) contain no designated root. A root enters only when one asks which unit-spaced graded score is represented. Once the residual affine structure is known, the score boundary at root q is simply λ_q .

Remark 3.3. Singleton states and repeated targets

A state with a singleton legal digit set contributes no adjacency equation; only the lower recursion applies. Adjacent digits are also allowed to have the same target. In that case (3.4) reduces to

$$U_n = N_r(n).$$

No branching or positivity assumption is needed for Theorem 3.1 itself.

4. Intrinsic rigidity

From this point onward assume that at least one accessible residual state branches. This is exactly the hypothesis that turns affine representability into a unique intrinsic place-value system.

Proposition 4.1. Iterated lower formula

For every state q and $n \geq 1$,

$$\lambda_q(n) = \sum_{j=0}^{n-1} a_{p^j(q)} U_{n-1-j}. \quad (4.1)$$

Equivalently, with

$$g_q(z) = \sum_{j \geq 0} a_{p^j(q)} z^j, \quad \Lambda_q(z) = \sum_{n \geq 0} \lambda_q(n) z^n,$$

one has

$$\Lambda_q(z) = z g_q(z) U(z). \quad (4.2)$$

Proof Iterate (3.3) from $\lambda_q(0) = 0$. The generating-function identity is the resulting convolution. $\square$

Theorem 4.2. Intrinsic Rigidity

Suppose the ordered residual automaton satisfies the interval condition and contains at least one genuine adjacency. Then any normalized scalar affine positional structure has a unique basis U . More strongly, any single genuine adjacency determines the only possible normalized candidate recursively. Once U is fixed, every λ_q is unique. Whenever the local compatibility equations hold, all U_n and $\lambda_q(n)$ are integers.

Proof Choose a genuine adjacency with targets r, s . Substituting (4.1) into (3.4) gives

$$U_n = N_r(n) + \sum_{j=0}^{n-1} (a_{p^j(r)} - a_{p^j(s)}) U_{n-1-j}. \quad (4.3)$$

The right-hand side depends only on $U_0, \dots, U_{n-1}$. Starting from $U_0 = 1$, (4.3) therefore determines U_n successively and uniquely. Formula (4.1) then determines every residual translation uniquely.

For integrality, use simultaneous induction. Initially $U_0 = 1$ and $\lambda_q(0) = 0$. If all translations at width n are integral, the chosen adjacency equation gives

$$U_n = N_r(n) + \lambda_r(n) - \lambda_s(n) \in \mathbb{Z}.$$

Then the lower equation gives

$$\lambda_q(n+1) = a_q U_n + \lambda_{p(q)}(n) \in \mathbb{Z}$$

for every q . $\square$

Corollary 4.3. Reference independence

If two genuine adjacencies generate different normalized candidates through (4.3), then no scalar affine positional structure exists. If positionality exists, every genuine adjacency generates the same intrinsic basis.

Definition 4.4. Intrinsic basis and residual boundaries

When the local compatibility equations hold, call the unique normalized sequence U the **intrinsic positional basis** of the ordered residual system. For each residual state q , define its **intrinsic boundary** by

$$\beta^{(q)} := \lambda_q. \quad (4.4)$$

A unit-spaced graded score rooted at q is represented positionally exactly when its boundary sequence equals $\beta^{(q)}$.

Remark 4.5. Branching is necessary for uniqueness

If every live state has exactly one legal digit, there are no adjacency equations. Any normalized sequence U may then be inserted into the lower recursions, producing an affine representation of the unique word in every residual cross-section. Thus branching is precisely what makes the positional basis intrinsic.

5. Lower-itinerary dynamics

Definition 5.1. Lower itinerary and lower-itinerary rank

For each state q , define its lower-threshold itinerary

$$\boxed{\iota(q) = (a_q, a_{p(q)}, a_{p^2(q)}, \dots).} \quad (5.1)$$

Write

$$q \sim_\ell r \iff \iota(q) = \iota(r).$$

Let

$$\boxed{b = |Q/\sim_\ell|.} \quad (5.2)$$

We call b the **lower-itinerary rank** of the ordered residual system.

The definition uses only the least-digit dynamics (Q, p, a) ; it does not assume positionality.

Proposition 5.2. Finite characterization

For $k \geq 0$, define equivalence relations recursively by

$$q \equiv_0 r \iff a_q = a_r,$$

and

$$q \equiv_{k+1} r \iff a_q = a_r \text{ and } p(q) \equiv_k p(r).$$

Then

$$q \equiv_k r \iff a_{p^j(q)} = a_{p^j(r)} \quad (0 \leq j \leq k).$$

The partitions stabilize after at most $s - 1$ strict refinements, and

$$\boxed{q \sim_\ell r \iff q \equiv_{s-1} r.} \quad (5.3)$$

Thus the first s lower thresholds always suffice to determine the infinite lower-itinerary classes.

Proof The prefix characterization follows by induction on k . The partitions refine monotonically. A strict refinement increases the number of classes, so at most $s - 1$ strict refinements are possible. Once a partition is stable, applying the deterministic refinement rule again leaves it unchanged forever. Hence the stable partition is precisely equality of infinite output itineraries. $\square$

Theorem 5.3. Lower-Itinerary Quotient

Assume intrinsic positionality. For states q, r , the following are equivalent:

$$q \sim_\ell r, \quad g_q = g_r, \quad \lambda_q = \lambda_r, \quad \beta^{(q)} = \beta^{(r)}. \quad (5.4)$$

Moreover,

$$q \sim_\ell r \implies a_q = a_r, \quad p(q) \sim_\ell p(r). \quad (5.5)$$

Hence the labelled lower dynamics descend to a functional quotient on the b itinerary classes.

Proof Equality of itineraries is exactly equality of the series g_q . By (4.2),

$$\Lambda_q = zg_qU.$$

Since $U_0 = 1$, $U(z)$ is invertible as a formal power series, so $\Lambda_q = \Lambda_r$ if and only if $g_q = g_r$. The boundary statement is merely $\beta^{(q)} = \lambda_q$. Finally, equality of itineraries forces equality of their first symbols and equality of their shifted tails, giving (5.5). $\square$

Remark 5.4. Boundary depth

For a state q , the first nonzero term of its lower itinerary determines the initial vanishing order of $\beta^{(q)}$. If

$$h(q) = \min\{j \geq 0 : a_{p^j(q)} > 0\},$$

then, when finite,

$$\text{ord}_z \mathcal{B}_q = h(q) + 1.$$

Because p acts on s states, either $h(q) \leq s - 1$ or $h(q) = \infty$. Boundary depth records an initial delay in the residual translation series; it should not be confused with Hankel rank or spectral complexity.

6. Compressed affine realization

The lower-itinerary quotient compresses translations, not censuses. Two residual states can have the same lower itinerary and therefore the same translation while still having different continuation counts.

Assume interval digit sets and at least one genuine adjacency. Choose a reference adjacency, construct the normalized candidate basis from (4.3), and define the candidate translations by (4.1). Let

$$\mathcal{B} = \{B_1, \dots, B_b\}$$

be the lower-itinerary classes and $\pi : Q \rightarrow \mathcal{B}$ the quotient map. Write

$$\mu_B(n) = \lambda_q(n) \quad (q \in B),$$

and collect these values in $\mu_n \in \mathbb{Q}^b$.

The lower map descends to the quotient. Let $\bar{P}$ be its functional matrix and $\bar{\mathbf{a}}$ the vector of class thresholds. Then

$$\boxed{\mu_{n+1} = \bar{\mathbf{a}}U_n + \bar{P}\mu_n.} \quad (6.1)$$

Choose a reference adjacency

$$e_* : d \rightarrow r_*, \quad d+1 \rightarrow s_*.$$

Set

$$c_* = e_{r_*} \in \mathbb{Q}^s, \quad h_* = e_{\pi(r_*)} - e_{\pi(s_*)} \in \mathbb{Q}^b.$$

Its tiling equation is

$$\boxed{U_n = c_*^T \mathbf{N}_n + h_*^T \mu_n.} \quad (6.2)$$

Now put

$$Y_n = \begin{pmatrix} \mathbf{N}_n \\ \mu_n \end{pmatrix} \in \mathbb{Q}^{s+b}.$$

The census recursion and (6.1)-(6.2) combine into

$$\boxed{Y_{n+1} = \bar{M}_* Y_n,} \quad (6.3)$$

with

$$\boxed{\bar{M}_* = \begin{pmatrix} A & 0 \\ \bar{\mathbf{a}}c_*^T & \bar{P} + \bar{\mathbf{a}}h_*^T \end{pmatrix}.} \quad (6.4)$$

The positional output is

$$\boxed{U_n = \bar{b}_*^T Y_n, \quad \bar{b}_*^T = (c_*^T, h_*^T).} \quad (6.5)$$

For any other adjacency e with targets r_e, s_e , define

$$\bar{b}_e^T = (e_{r_e}^T, e_{\pi(r_e)}^T - e_{\pi(s_e)}^T). \quad (6.6)$$

The candidate is intrinsically positional exactly when

$$\boxed{\bar{b}_e^T Y_n = \bar{b}_*^T Y_n \text{ for every adjacency } e \text{ and every } n.} \quad (6.7)$$

Theorem 6.1. Compressed Candidate Realization and Recognition

Let an ordered residual automaton have s residual states, interval digit sets, at least one genuine adjacency, and lower-itinerary rank b . Construct the candidate system above from any reference adjacency. Then:

1. the candidate basis and candidate residual translations are rational:

$$\boxed{U(z) = \bar{b}_*^T (I - z\bar{M}_*)^{-1} Y_0,} \quad (6.8)$$

$$\boxed{\Lambda_q(z) = d_q^T (I - z\bar{M}_*)^{-1} Y_0,} \quad (6.9)$$

where $d_q^T = (0, e_{\pi(q)}^T)$;

2. if all compatibility equations (6.7) hold, then the candidate is the intrinsic positional system and

$$\boxed{r_U \leq s + b - 1, \quad r_{\beta(q)} \leq s + b;} \quad (6.10)$$

3. if compatibility fails, some adjacency detects failure by continuation depth at most

$$\boxed{s + b - 2.} \quad (6.11)$$

Proof Rationality is immediate from the finite realization. The only point requiring explanation is the loss of one dimension from positional value.

Let

$$\boxed{\bar{g} = \begin{pmatrix} 0 \\ \mathbf{1}_b \end{pmatrix}.} \quad (6.12)$$

Since $\bar{P}\mathbf{1}_b = \mathbf{1}_b$ and $h_*^T \mathbf{1}_b = 0$,

$$\boxed{\bar{M}_* \bar{g} = \bar{g}.} \quad (6.13)$$

Every adjacency output also annihilates this vector:

$$\boxed{\bar{b}_e^T \bar{g} = 0.} \quad (6.14)$$

Thus positional value factors through the quotient by $\langle \bar{g} \rangle$. If $\bar{g}$ is unreachable, the reachable space already has dimension at most $s + b - 1$; if it is reachable, it is unobservable. Either way,

$$r_U \leq s + b - 1.$$

A residual boundary is different. Its output functional d_q satisfies

$$d_q^T \bar{g} = 1,$$

so the gauge mode can remain visible and only the full bound $s + b$ is universal.

For recognition, let

$$f_e^T = \bar{b}_e^T - \bar{b}_*^T.$$

Then $f_e^T \bar{g} = 0$, so the discrepancy

$$\Delta_e(n) = f_e^T \bar{M}_*^n Y_0$$

has realization dimension at most $s + b - 1$. If its first $s + b - 1$ values vanish, for $0 \leq n \leq s + b - 2$, the finite Krylov space forces $\Delta_e \equiv 0$. Hence any failure appears by that depth. $\square$

6.1 Exact realization complexity

Let

$$\mathcal{K} = \text{span}\{Y_0, \bar{M}_* Y_0, \bar{M}_*^2 Y_0, \dots\}$$

be the reachable Krylov space. For positional value let

$$\mathcal{O}_U = \{x \in \mathcal{K} : \bar{b}_*^T \bar{M}_*^j x = 0 \text{ for all } j \geq 0\}.$$

Then

$$\boxed{r_U = \dim(\mathcal{K}/\mathcal{O}_U) = \text{rank } H_U.} \quad (6.15)$$

For a residual boundary,

$$\boxed{r_{\beta(q)} = \dim(\mathcal{K}/\mathcal{O}_{\beta,q}) = \text{rank } H_{\beta(q)}.} \quad (6.16)$$

We use Hankel rank as the definition of scalar recurrence complexity. For a reduced strictly proper rational function P/Q , it equals $\deg Q$, with repeated poles counted by multiplicity.

6.2 Finite recognition algorithm

The theorem gives a finite test:

1. compute the accessible minimal live residual DFA and its state count s ;
2. verify interval digit sets and extract $a_q, p(q)$;
3. compute the lower-itinerary partition and its size b ;
4. choose one genuine adjacency and build the candidate basis and compressed realization;
5. test every remaining adjacency through continuation depth $s + b - 2$.

If all tests pass, Theorem 3.1 gives intrinsic affine positionality at every width. Exact basis, boundary and Hankel-rank calculations then follow from the same finite realization.

Remark 6.2. Recognition sharpness

The dimension bound $s + b - 2$ is attained for many parameter pairs, but it is not universally sharp over the integer digit lattice. The strict cell $(4, 4)$ is treated next. This does not affect the sharp recurrence-complexity bounds of Section 7: recognition asks how long an incompatible automaton can mimic the candidate affine system, whereas Section 7 concerns the Hankel rank of genuinely positional systems.

6.3 Exact recognition radius and an arithmetic defect

For $K = \mathbb{Z}$, let $H_{\mathbb{Z}}(s, b)$ denote the largest first incompatibility depth among rejecting interval-legal integer-digit systems with s live residual states and lower-itinerary rank b . Define $H_{\mathbb{Q}}(s, b)$ on the corresponding rational affine relaxation:

the ordered transition-target skeleton is kept fixed, but the least-threshold vector in the affine compatibility equations is allowed to range over $\mathbb{Q}$. The rational relaxation need not itself be a language over a rational digit alphabet; it is used only to identify arithmetic obstructions.

Theorem 6.1 gives

$$H_K(s, b) \leq s + b - 2. \quad (6.17)$$

Theorem 6.3. A strict integer recognition radius

$$\boxed{H_{\mathbb{Z}}(4, 4) = 5.} \quad (6.18)$$

Proof For the lower bound, consider the four-state automaton

$$\begin{aligned} q_0 : 0 &\rightarrow q_0, & 1 &\rightarrow q_2, \\ q_1 : 0 &\rightarrow q_2, \\ q_2 : 0 &\rightarrow q_3, & 1 &\rightarrow q_0, \\ q_3 : 1 &\rightarrow q_1. \end{aligned} \quad (6.18a)$$

Its four lower itineraries are distinct. Taking the adjacency at q_0 as reference gives candidate basis

$$U = (1, 2, 3, 5, 9, 15, \dots),$$

and the competing discrepancy has generating function

$$\Delta(z) = -\frac{z^5(1+z)}{(z^3 - z^2 - 1)(z^4 - z^3 - z^2 - z + 1)}. \quad (6.18b)$$

Hence $\Delta_0 = \dots = \Delta_4 = 0$ and $\Delta_5 \neq 0$, so

$$H_{\mathbb{Z}}(4, 4) \geq 5.$$

It remains to exclude first failure at depth 6. Any such witness must have full four-dimensional census Krylov space; otherwise the census contributes at most three modes and the translation quotient at most three, so every discrepancy has realization dimension at most six and cannot have six initial zeros followed by a nonzero term.

Form the simple adjacency-target graph G : an edge $r \rightarrow s$ records two consecutive legal digits with targets r, s . Compatibility through four widths and Proposition 5.2 imply that G has outdegree at most one. Indeed, two edges $r \rightarrow s$ and $r \rightarrow t$ would force the first four lower-threshold itinerary symbols of s and t to agree, hence their complete lower itineraries to agree, contradicting $b = 4$.

If G is acyclic, the exact finite classification of Section 13.4 shows that depth-six compatibility is always implied by the preceding compatibility equations.

Suppose instead that G contains a directed cycle. For any signed circulation z on G , summing the corresponding adjacency equations cancels the translations and gives

$$w_z^T N(n) = L_z U_n. \quad (6.19)$$

Full census rank therefore forces all directed cycles to have the same normalized occupation vector. Since G is functional, it follows that G has exactly one directed cycle.

When the cyclic component contains all states, the adjacency successor defines a functional matrix T . With $D = I - T$, $F_n = N(n) + D\lambda(n)$, and $C = DP - AD$, compatibility gives $F_n = U_n \mathbf{1}$ and

$$F_{n+1} = U_1 U_n \mathbf{1} + C\lambda(n). \quad (6.20)$$

Successive compatible widths force $Ca, CPa, CP^2a, CP^3a \in \langle \mathbf{1} \rangle$; Cayley-Hamilton for the four-state lower matrix P propagates this to all powers and hence to every later width.

With exactly two weak components, there is only one relative component-translation offset after quotienting the common gauge. Let h span this one-dimensional component-constant quotient. All within-component translation differences are already determined by the current adjacency equations. Under the lower recursion, the only possible hidden contribution at the next width is the class of Ph . If every next-width adjacency output annihilates this contribution, then Ph is again component-constant modulo the common gauge, hence represents a scalar multiple of h ; induction shows that the hidden mode is unobservable at every later width. Otherwise some next-width compatibility equation has a nonzero coefficient on the hidden offset and reconstructs it linearly from the census data. In that case four independent census vectors force an exact intertwining relation between census and translation dynamics, so compatibility again propagates.

The remaining three-component cyclic graphs form finite symbolic families. For a two-cycle, exact elimination gives 192 parity/start templates. In 176 templates the depth-six equation lies in the ideal generated by depths one through five. The remaining 16 templates are one family up to symmetry: compatibility forces one core row to have length one, the other to have even length $2m$ with $m \geq 2$ in the genuine two-cycle case, and a threshold difference

$$m - \frac{1}{2},$$

which is impossible for integer digits. The two three-component self-loop types are likewise eliminated exactly: the feeder form reduces to six lower-map types up to symmetry, while the disconnected-edge form has 168 symbolic templates; in every integer-feasible case depth-six compatibility is implied by the earlier equations. Section 13.4 records the exact verification protocol and counts.

Thus no integer-digit $(4, 4)$ system first fails at depth six, and the explicit depth-five witness proves (6.18). $\square$

Proposition 6.4. Rational sharpness and arithmetic origin

For the rational-threshold relaxation,

$$\boxed{H_{\mathbb{Q}}(4, 4) = 6.} \quad (6.21)$$

Proof The general upper bound gives $H_{\mathbb{Q}}(4, 4) \leq 6$. In the exceptional two-cycle family from the proof above, compatibility through depth five forces

$$2(a_i - a_j) = 2m - 1, \quad (6.22)$$

with $m \geq 2$ for a genuine two-cycle. This has rational solutions but no integer solutions. For example, at $m = 2$ one representative ordered target skeleton, in state order (q_0, q_1, x, y) , has rows

$$(x), \quad (q_0), \quad (y), \quad (q_0, q_1, q_0, q_1),$$

lower map $p = (x, q_0, y, q_0)$, and threshold vector $a = (0, 0, 0, -3/2)$. Its four rational lower itineraries are distinct. The depth-six discrepancy is

$$\Delta_6 = \pm \frac{2m - 1}{2} \neq 0, \quad (6.23)$$

so at $m = 2$ its magnitude is $3/2$. Hence the rational relaxation attains the ambient realization bound, while the integer digit lattice removes the last mode. $\square$

The distinction between $H_{\mathbb{Q}}$ and $H_{\mathbb{Z}}$ will be useful below: finite recognition complexity is controlled not only by reachable-observable dimension but also by arithmetic feasibility of the local threshold equations.

7. Sharp two-parameter recurrence complexity

For an admissible pair (s, b) , let $R_U(s, b)$ be the largest Hankel rank of an intrinsic normalized basis among minimal accessible right-prolongable intrinsically positional systems with s residual states and lower-itinerary rank b . Define $R_{\beta}(s, b)$ similarly for residual boundaries.

Theorem 6.1 gives

$$R_U(s, b) \leq s + b - 1, \quad R_{\beta}(s, b) \leq s + b.$$

We now show that both bounds are attained for every $1 \leq b \leq s$.

7.1 Extremal family for $2 \leq b \leq s$

Fix

$$2 \leq b \leq s, \quad m = s - b + 1.$$

Use states

$$q_0, \dots, q_{b-2}, \quad r_0, \dots, r_{m-1}.$$

The q_i form a deterministic threshold tail:

$$q_i : i \rightarrow q_{i+1} \quad (0 \leq i < b-2), \quad (7.1)$$

$$q_{b-2} : b-2 \rightarrow r_0. \quad (7.2)$$

The r_j form a lower cycle:

$$r_j : b-1 \rightarrow r_{j+1} \quad (0 \leq j < m-1), \quad (7.3)$$

and the final state branches:

$$\boxed{r_{m-1} : b-1 \rightarrow r_0, \quad b \rightarrow q_0.} \quad (7.4)$$

There is only one genuine adjacency, so once its equation defines the candidate basis, Theorem 3.1 supplies the full affine representation.

The tail states have distinct lower itineraries, while every r_j has the constant itinerary $(b-1)^\infty$. Hence the lower-itinerary rank is exactly b .

The automaton is strongly connected and minimal. Tail states have different singleton legal digit sets. Cycle states are distinguished by the shortest continuation containing the unique upper digit b : from r_j its length is $m-j$.

7.2 Generating functions and basis sharpness

Take r_0 as the reference census state. There are two return routes to r_0 : the lower cycle of length m and the upper excursion of length s . Therefore

$$\boxed{C(z) = \frac{P_s(z)}{1 - z^m - z^s},} \quad (7.5)$$

where $P_k(z) = 1 + z + \dots + z^{k-1}$.

The cycle itinerary series is

$$g_r(z) = \frac{b-1}{1-z},$$

while

$$g_{q_0}(z) = \frac{z(1-z^{b-1})}{(1-z)^2}.$$

Their difference is

$$\boxed{H_b(z) = (b-1) + (b-2)z + \dots + z^{b-2}.} \quad (7.6)$$

The unique adjacency gives $U = C + \lambda_r - \lambda_{q_0}$, hence

$$\boxed{U(z) = \frac{P_s(z)}{(1 - z^m - z^s)K_b(z)},} \quad (7.7)$$

with

$$\boxed{K_b(z) = 1 - (b-1)z - (b-2)z^2 - \dots - z^{b-1}.} \quad (7.8)$$

The denominator has degree $s + b - 1$. It remains only to rule out cancellation.

Lemma 7.1. Noncancellation

The numerator $P_s(z)$ is coprime to both denominator factors in (7.7).

Proof Let $\zeta \neq 1$ be a zero of P_s . Then $\zeta^s = 1$, so

$$1 - \zeta^m - \zeta^s = -\zeta^m \neq 0.$$

Thus the census factor does not cancel.

For K_b , use

$$\boxed{(1-z)^2 K_b(z) = 1 - (b+1)z + (b+1)z^2 - z^{b+1}.} \quad (7.9)$$

If $|\zeta| = 1$, $\zeta \neq 1$, and $K_b(\zeta) = 0$, then

$$1 + \zeta + \dots + \zeta^b = (b+1)\zeta.$$

The right side has modulus $b+1$. Equality in the triangle inequality on the left is possible only when all summands have the same argument, which forces $\zeta = 1$, a contradiction. Hence K_b has no nontrivial unit-circle zero and cannot cancel P_s . $\square$

Therefore (7.7) is reduced and strictly proper, so

$$\boxed{r_U = s + b - 1.} \quad (7.10)$$

7.3 Boundary sharpness

Choose q_0 as root. Since

$$g_{q_0}(z) = \frac{zP_{b-1}(z)}{1-z},$$

we obtain

$$\boxed{\mathcal{B}_{q_0}(z) = \frac{z^2 P_{b-1}(z) P_s(z)}{(1-z)(1-z^m - z^s) K_b(z)}.} \quad (7.11)$$

The denominator has degree $s + b$. No new cancellation occurs. If $P_{b-1}(\zeta) = 0$, then $\zeta^{b-1} = 1$, $\zeta \neq 1$, and because $s = m + b - 1$,

$$1 - \zeta^m - \zeta^s = 1 - 2\zeta^m \neq 0.$$

Also

$$K_b(\zeta) = \frac{1 - b\zeta}{1 - \zeta} \neq 0,$$

and $P_{b-1}(1) = b - 1 \neq 0$. Hence

$$\boxed{r_{\beta(q_0)} = s + b.} \quad (7.12)$$

For $b = 2$, $K_2(z) = 1 - z$, so the boundary has a genuine double pole at $z = 1$; the degree count is unchanged.

7.4 The endpoint $b = 1$ for $s \geq 2$

Take an s -state lower cycle

$$q_j : 1 \rightarrow q_{j+1} \quad (0 \leq j < s-1),$$

with

$$q_{s-1} : 1 \rightarrow q_0, \quad 2 \rightarrow q_1.$$

Every state has itinerary 1^∞ , so $b = 1$, while the states remain distinct by their distances to the unique upper digit 2. The branch targets have identical translations, hence

$$U(z) = \frac{P_{s-1}(z)}{1 - z^{s-1} - z^s}, \quad r_U = s.$$

Moreover

$$\mathcal{B}_{q_0}(z) = \frac{zP_{s-1}(z)}{(1-z)(1-z^{s-1}-z^s)},$$

which is reduced of denominator degree $s + 1$.

7.5 The one-state endpoint

For $(s, b) = (1, 1)$, take one state q with loops

$$q : 1 \rightarrow q, \quad 2 \rightarrow q.$$

Then

$$U_n = 2^n, \quad U(z) = \frac{1}{1-2z},$$

so $r_U = 1$. Also

$$\mathcal{B}_q(z) = \frac{z}{(1-z)(1-2z)},$$

so $r_{\beta(q)} = 2$.

Theorem 7.2. Sharp Two-Parameter Complexity

For every admissible pair $1 \leq b \leq s$,

$$\boxed{R_U(s, b) = s + b - 1,} \quad (7.13)$$

$$\boxed{R_\beta(s, b) = s + b.} \quad (7.14)$$

Proof The upper bounds are Theorem 6.1. Sections 7.1-7.5 provide minimal strongly connected constructions attaining them. $\square$

Remark 7.3. Interpretation

At maximal complexity, the census contributes s modes and the lower-itinerary translations contribute b . A residual boundary can observe all $s + b$ modes. Positional value cannot see the common-translation gauge, so its maximum is $s + b - 1$. The earlier state-count-only value $2s - 1$ is the special case $b = s$.

8. Twisted coboundaries and residual transfer

The local tiling equations have an equivalent edge-defect formulation that is useful for path integration and root change.

For a legal edge

$$e : q \xrightarrow{a} r,$$

define its width- n skip count

$$S_e(n) = \sum_{\substack{d \in D(q) \\ d < a}} N_{\delta(q,d)}(n). \quad (8.1)$$

Thus $S_e(n)$ is the number of width- n continuations contained in lexicographically earlier digit cylinders at the source state. Define the defect sequence

$$\boxed{D_e = S_e - aU.} \quad (8.2)$$

Let T denote forward shift on sequences:

$$(Tx)_n = x_{n+1}.$$

Theorem 8.1. Twisted Coboundary Characterization

Under interval legality, residual affine positionality is equivalent to the existence of state potentials λ_q , $\lambda_q(0) = 0$, satisfying

$$\boxed{D_e = \lambda_r - T\lambda_q} \quad (8.3)$$

for every legal edge $e : q \xrightarrow{a} r$.

After lower-itinerary compression this becomes

$$\boxed{D_e = \mu_{\pi(r)} - T\mu_{\pi(q)}.} \quad (8.4)$$

Proof For $w \in L_r(n)$, rank decomposes as

$$R_q^{(n+1)}(aw) = S_e(n) + R_r^{(n)}(w).$$

Substitute the residual affine formula and cancel $V_U(w)$:

$$S_e(n) - aU_n = \lambda_r(n) - \lambda_q(n+1),$$

which is (8.3).

Conversely, take the lower edge $e_0 = (q, a_q, p(q))$. Its skip count is zero, so (8.3) gives

$$T\lambda_q = a_q U + \lambda_{p(q)},$$

the lower recursion. For adjacent legal digits $d, d+1$ with targets r, s , the skip-count difference is N_r . Subtracting their two coboundary equations gives

$$N_r - U = \lambda_s - \lambda_r,$$

which is the adjacency tiling equation. Theorem 3.1 then gives residual affine positionality. Compression follows from $\lambda_q = \mu_{\pi(q)}$. $\square$

8.1 Path integration

For a legal path

$$\gamma = e_1 e_2 \cdots e_k$$

from q to r , define

$$D_\gamma = \sum_{j=1}^k T^{k-j} D_{e_j}. \quad (8.5)$$

The edge coboundaries telescope to

$$D_\gamma = \lambda_r - T^k \lambda_q. \quad (8.6)$$

Equivalently, in compressed coordinates,

$$D_\gamma = \mu_{\pi(r)} - T^k \mu_{\pi(q)}. \quad (8.7)$$

If q is taken as root, its intrinsic boundary is $\beta^{(q)} = \lambda_q$. Hence a path $\gamma : q \rightarrow r$ of length k gives the exact root-change law

$$\beta^{(r)} = T^k \beta^{(q)} + D_\gamma. \quad (8.8)$$

This identity is path-independent because both sides equal λ_r .

For a cycle returning to the same lower-itinerary class B ,

$$D_\gamma = (I - T^k) \mu_B. \quad (8.9)$$

Cycle defects therefore need not vanish. Vanishing is a special boundary condition, not a universal property.

Corollary 8.2. Residual heredity

Every residual language inherits the same positional basis U and the corresponding residual affine rank formula. If a residual state q can still reach a branching state, then the re-rooted accessible residual automaton determines U intrinsically.

Let s_q be the number of states reachable from q , and let b_q be the number of lower-itinerary classes represented among those states. Whenever branching remains reachable,

$$\boxed{r_U \leq s_q + b_q - 1.} \quad (8.10)$$

In particular, if the original system is complexity-extremal with $r_U = s + b - 1$, then every state from which branching is reachable must satisfy

$$s_q = s, \quad b_q = b.$$

9. Fixed-width and genealogical root boundaries

The intrinsic theory is root-free. A particular ranking convention is positionally represented only when its prescribed grade boundary agrees with the intrinsic boundary of the chosen root.

9.1 Fixed-width rank

Fixed-width rank uses the score

$$G_n(w) = R_{q_*}^{(n)}(w),$$

so its boundary is identically zero.

Proposition 9.1. Fixed-width criterion

The chosen root is fixed-width rank positional if and only if

$$\boxed{\beta^{(q_*)} = \lambda_{q_*} \equiv 0.} \quad (9.1)$$

Equivalently,

$$\boxed{a_{p^j(q_*)} = 0 \quad \text{for all } j \geq 0.} \quad (9.2)$$

Proof The first equivalence is the definition of the fixed-width boundary. From (4.2),

$$\Lambda_{q_*}(z) = z g_{q_*}(z) U(z).$$

Since $U_0 = 1$, $U(z)$ is formally invertible. Thus $\Lambda_{q_*} = 0$ if and only if $g_{q_*} = 0$, which is exactly (9.2). $\square$

This is the infinite-depth lower-boundary regime: the least-digit orbit of the root never encounters a positive threshold. The intrinsic basis is nevertheless determined by a genuine branching adjacency elsewhere in the accessible residual system.

9.2 Genealogical rank

Let

$$C_n = N_{q_*}(n)$$

be the root census and define

$$B_n = \sum_{m=0}^{n-1} C_m, \quad B_0 = 0. \quad (9.3)$$

The genealogical rank of a root word $w \in L(n)$ is

$$G(w) = B_n + R_{q_*}^{(n)}(w).$$

Thus genealogical positionality is precisely the root-boundary condition

$$\boxed{\beta^{(q_*)} = B.} \quad (9.4)$$

Proposition 9.2. Root digit and virtual adjacency

If the chosen root is genealogically positional, then

$$\boxed{a_{q_*} = 1.} \quad (9.5)$$

Moreover,

$$\boxed{U_n = C_n + \lambda_{q_*}(n) - \lambda_{p(q_*)}(n).} \quad (9.6)$$

Proof Since $B_1 = C_0 = 1$, width one gives

$$a_{q_*} = \lambda_{q_*}(1) = B_1 = 1.$$

The lower recursion at the root is therefore

$$B_{n+1} = U_n + \lambda_{p(q_*)}(n).$$

But

$$B_{n+1} = B_n + C_n = \lambda_{q_*}(n) + C_n.$$

Rearranging gives (9.6). $\square$

Equation (9.6) has exactly the same affine form as a genuine adjacent-cylinder equation, even though it is induced by the cumulative grading boundary rather than by two adjacent root digits. We call it the **virtual root adjacency**.

Proposition 9.3. Genealogical generating function

Let

$$g_*(z) = g_{q_*}(z) = \sum_{j \geq 0} a_{p^j(q_*)} z^j.$$

If the chosen root is genealogically positional, then

$$\boxed{U(z) = \frac{C(z)}{(1-z)g_*(z)}.} \quad (9.7)$$

Proof From (4.2),

$$\Lambda_{q_*}(z) = zg_*(z)U(z).$$

Under genealogical positionality $\Lambda_{q_*} = B$, while

$$B(z) = \frac{zC(z)}{1-z}.$$

Cancel z . Since $a_{q_*} = 1$, $g_*(0) = 1$, so formal division is valid. $\square$

9.3 Mutual exclusivity at a fixed root

For an infinite prefix-closed language,

$$B_1 = 1.$$

Hence a chosen root cannot be simultaneously fixed-width and genealogically positional under the same digit labels and normalization: the intrinsic root boundary cannot be both 0 and B .

This does not contradict residual heredity. The intrinsic affine basis is inherited under re-rooting, but fixed-width and genealogical type are root-specific properties of the intrinsic boundary selected by that root.

10. Positive bases and Perron growth

The intrinsic affine theory allows zero, negative and oscillatory place values. Growth statements therefore require positivity.

Let A be the census matrix of the accessible live residual DFA and write $\rho = \rho(A)$.

Lemma 10.1. Census growth

For the root census $C_n = e_{q_*}^T A^n \mathbf{1}$,

$$\boxed{\lim_{n \rightarrow \infty} C_n^{1/n} = \rho(A).} \quad (10.1)$$

Proof The upper bound is Gelfand's formula. For the lower bound, choose an accessible strongly connected component whose irreducible block B has spectral radius $\rho(A)$. A fixed path from the root reduces the problem to $e_i^T B^n \mathbf{1}$.

If B has period d , split it into cyclic classes. On each class, B^d is primitive with spectral radius $\rho(A)^d$. Perron-Frobenius theory on each residue class modulo d gives

$$(e_i^T B^n \mathbf{1})^{1/n} \rightarrow \rho(A).$$

A fixed access-path shift does not change the exponential rate. $\square$

Proposition 10.2. Universal lower growth bound

If the intrinsic basis is positive, then

$$\boxed{\rho(A) \leq \limsup_{n \rightarrow \infty} U_n^{1/n}.} \quad (10.2)$$

Proof At width n , the root values form an interval of length C_n . Because the alphabet is finite, its span is at most a constant multiple of $\sum_{j < n} U_j$. Hence

$$C_n - 1 \leq M_\Sigma \sum_{j < n} U_j.$$

Taking exponential rates and using Lemma 10.1 gives the claim. $\square$

The reverse inequality is false in the general intrinsic theory.

Example 10.3. Positive super-Perron intrinsic basis

For the sharp family with $s = b = 3$,

$$C(z) = \frac{1 + z + z^2}{1 - z - z^3}, \quad U(z) = \frac{1 + z + z^2}{(1 - z - z^3)(1 - 2z - z^2)}.$$

The basis coefficients are positive. The census grows at the positive root of

$$x^3 - x^2 - 1 = 0,$$

approximately 1.46557, while $1 - 2z - z^2$ gives basis growth rate

$$1 + \sqrt{2} \approx 2.41421.$$

Thus a positive intrinsic basis can grow strictly faster than the language census.

Theorem 10.4. Perron growth for canonical root boundaries

If a positive intrinsic basis is fixed-width positional or genealogically positional at the chosen root, then

$$\boxed{\limsup_{n \rightarrow \infty} U_n^{1/n} = \rho(A).} \quad (10.3)$$

If U is eventually nondecreasing, then

$$\boxed{\lim_{n \rightarrow \infty} U_n^{1/n} = \rho(A).} \quad (10.4)$$

Proof Proposition 10.2 gives the lower bound. It remains to bound U from above.

For fixed-width positionality, every threshold on the root lower orbit is zero. Let $q_h = p^h(q_*)$ be the first branching state on that orbit. At q_h , the adjacent digits 0, 1 give

$$U_n = N_{p(q_h)}(n) - \lambda_s(n) \leq N_{p(q_h)}(n),$$

because positivity of U and nonnegative thresholds imply $\lambda_s(n) \geq 0$. Prefixing the $h + 1$ lower edges gives

$$U_n \leq C_{n+h+1}.$$

Lemma 10.1 yields the upper exponential bound.

For genealogical positionality, $a_{q_*} = 1$ and

$$B_n = \lambda_{q_*}(n) = U_{n-1} + \sum_{j=1}^{n-1} a_{p^j(q_*)} U_{n-1-j}.$$

All terms are nonnegative, so

$$U_{n-1} \leq B_n.$$

The cumulative census has the same exponential rate as C_n , giving the required upper bound. The case $\rho(A) = 1$ is immediate for positive integral U .

Finally, if U is eventually nondecreasing, then

$$\sum_{j < n} U_j \leq O(nU_{n-1}).$$

Together with the census-span inequality, this forces the liminf and limsup of $U_n^{1/n}$ to agree at $\rho(A)$. $\square$

11. Examples

The preceding theory separates three kinds of structure that are easily conflated in concrete numeration systems: the residual census dynamics, the lower-threshold dynamics, and the root boundary. The following examples illustrate that separation.

11.1 A fixed-width positional system with nontrivial translations

Consider the two-state ordered automaton with states A, B , root A , and transitions

$$\begin{aligned} A : 0 &\rightarrow A, & 1 &\rightarrow B, & 2 &\rightarrow B, \\ B : 1 &\rightarrow A, & 2 &\rightarrow B. \end{aligned}$$

The legal digit sets are intervals. The lower map is

$$p(A) = A, \quad p(B) = A,$$

with lower thresholds

$$a_A = 0, \quad a_B = 1.$$

Thus the root lower itinerary is identically zero, so the intrinsic root boundary is zero and the system is fixed-width positional when its adjacency tiling equations hold.

The resulting normalized basis begins

$$1, 2, 5, 12, 29, \dots,$$

and satisfies the Pell recurrence

$$U_{n+2} = 2U_{n+1} + U_n.$$

The translation at A vanishes, while the translation at B is nonzero. This illustrates that fixed-width positionality requires only the chosen root boundary to vanish; other residual boundaries may remain nontrivial.

11.2 Genealogical positionality and the virtual root adjacency

Take three states q_0, q_1, q_2 , with root q_0 , and transitions

$$q_0 : 1 \rightarrow q_1,$$

$$q_1 : 0 \rightarrow q_2,$$

$$q_2 : 1 \rightarrow q_0, \quad 2 \rightarrow q_1.$$

There is one genuine adjacency, at q_2 . The lower-threshold orbit from the root is periodic with word

$$1, 0, 1,$$

so

$$g_{q_0}(z) = \frac{1 + z^2}{1 - z^3}.$$

The root census is

$$C(z) = \frac{1 + z}{1 - z^2 - z^3}.$$

For genealogical positionality, Proposition 9.3 gives

$$U(z) = \frac{C(z)}{(1 - z)g_{q_0}(z)} = \frac{(1 + z)(1 + z + z^2)}{(1 + z^2)(1 - z^2 - z^3)}.$$

The basis begins

$$1, 2, 2, 2, 3, 5, 6, 7, 10, 14, \dots$$

Here the root offset is not zero. It is the cumulative census

$$B_n = \sum_{j < n} C_j,$$

and the root digit 1 supplies the virtual root adjacency that determines U .

11.3 Same intrinsic basis, different residual boundaries

Let an intrinsically positional automaton have residual states q, r with distinct lower-threshold itineraries. Then

$$\mathcal{B}_q(z) = zg_q(z)U(z), \quad \mathcal{B}_r(z) = zg_r(z)U(z).$$

The same place-value basis therefore supports different residual grading boundaries. If a path $\gamma : q \rightarrow r$ has length k , Theorem 8.1 gives

$$\beta^{(r)} = T^k \beta^{(q)} + D_\gamma.$$

This is the precise sense in which positionality is residual-hereditary while fixed-width or genealogical type is root-specific.

11.4 The two-parameter extremal family

Fix $2 \leq b \leq s$ and put

$$m = s - b + 1.$$

Use the tail-plus-cycle automaton from Section 7. Its lower-itinerary quotient has exactly b classes, while the full minimal residual DFA has s states. The positional generating function is

$$U(z) = \frac{P_s(z)}{(1 - z^m - z^s)K_b(z)},$$

where

$$K_b(z) = 1 - (b-1)z - (b-2)z^2 - \dots - z^{b-1}.$$

The reduced denominator has degree

$$s + b - 1,$$

so the positional Hankel rank attains the universal bound. With the root at the beginning of the threshold tail, the boundary generating function has reduced denominator degree

$$s + b.$$

Thus this single family realizes the complete sharp pair

$$r_U = s + b - 1, \quad r_\beta = s + b.$$

11.5 A positive intrinsic basis with super-Perron growth

The example from Section 10 also illustrates why the general intrinsic theory is strictly broader than either canonical rank convention. For $s = b = 3$,

$$U(z) = \frac{1 + z + z^2}{(1 - z - z^3)(1 - 2z - z^2)}$$

has positive coefficients but grows at rate $1 + \sqrt{2}$, strictly faster than the census Perron rate. Hence the root boundary can carry positional growth that is invisible to the language census. Fixed-width and genealogical boundaries exclude this phenomenon by additional root constraints.

12. Relation to abstract numeration systems and positional numeration

Abstract numeration systems enumerate the words of an infinite regular language, usually in genealogical order, and use that enumeration to represent the nonnegative integers. Lecomte and Rigo developed this regular-language framework and its recognizability theory [1,2]. Recent work on Dumont-Thomas systems studies when an important subclass admits a positional realization [4], while other work examines regular positional systems outside the classical dominant-root setting [5].

The present paper starts from a different datum: fixed-width lexicographic ranks of residual languages. The global ranking convention enters only afterwards through the root boundary. This is why fixed-width and genealogical rank share the same local affine theory even though their global ranks differ.

The assumptions of prefix closure and right-prolongability are substantive. They ensure that every live residual supports every continuation width, so one place-value sequence and one shift operator can govern the whole residual system synchronously. For a general regular language, residuals may have different admissible-width sets, and this synchronous formulation no longer applies directly.

The lower-itinerary parameter b is also specific to the present problem. It counts distinct least-digit threshold streams and separates translation complexity from the s -state census dynamics. The sharp law

$$R_U(s, b) = s + b - 1$$

therefore refines state-count-only estimates in a way ordinary census enumeration does not see.

Finally, the compressed affine realization connects the theory with standard rational-series methods. Every intrinsic basis and boundary is rational, and its Hankel rank is the dimension of the appropriate reachable-observable quotient. The lower-itinerary quotient is a canonical reduction performed before ordinary realization minimization.

Extending this theory to arbitrary abstract numeration systems will require additional structure for admissible widths or phases. That is the main obstruction left by the present hypotheses.

13. Computational verification

The proofs in this paper are structural and do not depend on computation. Nevertheless, exact finite computation is useful for regression testing the formulas and for exploring sharpness questions not settled by the main theorems.

13.1 Exact rank enumeration

For small automata, the following quantities can be checked directly:

1. enumerate $L_q(n)$ for each residual state and small width n ;
2. compute zero-based lexicographic ranks;
3. construct the candidate basis from one reference adjacency;
4. construct all residual translations from the lower recursion;
5. verify

$$R_q^{(n)}(w) = V_U(w) - \lambda_q(n)$$

word by word;

6. compare all adjacency tiling equations.

These checks provide a direct finite verification of the root-free Local Affine Characterization on concrete examples.

13.2 Lower-itinerary refinement

The lower-itinerary rank b is computed independently of positionality. Beginning with the partition by lower threshold a_q , repeatedly refine states according to the pair

$$(a_q, [p(q)]).$$

The process stabilizes after at most $s - 1$ strict refinements. The resulting number of classes agrees with equality of the first s lower-threshold symbols and, under positionality, with equality of the computed residual boundary sequences.

13.3 Hankel-rank checks

For a compressed realization of dimension $d = s + b$, exact rational arithmetic can form

$$H_d = (U_{i+j})_{0 \leq i, j < d}$$

and the analogous boundary Hankel block. The positional rank is bounded by $d - 1$ because of the gauge quotient, while the boundary may attain d . For the extremal family of Section 7, exact ranks reproduce

$$r_U = s + b - 1, \quad r_\beta = s + b.$$

These checks avoid numerical root finding and are robust against repeated poles or nilpotent transients.

13.4 Exact verification of the $(4, 4)$ recognition radius

Theorem 6.3 uses finite exact algebra only after the graph-theoretic reductions have reduced the problem to small structural families. No floating-point arithmetic or bounded search over threshold values is used.

For the acyclic case, every transition-target row is a nonempty simple path on four states. Enumerating the complete four-row universe and retaining exactly

the systems whose union is a functional DAG with at least two genuine adjacency edges gives

$$216,360$$

row structures. Exact affine elimination over $\mathbb{Q}$ leaves

$$63,240$$

structures compatible through depth five. In every one, each depth-six compatibility equation lies in the affine row span of the depth-one through depth-five equations. Thus there are zero acyclic depth-six witnesses.

For the three-component two-cycle case, splitting by source placement, starting phase and row parity gives 192 symbolic templates. Exact ideal reduction yields 176 generic templates in which Δ_6 is already generated by $\Delta_1, \dots, \Delta_5$, plus 16 symmetric exceptional templates. Their reduced equations force one core row to have length one, the other length $2m$, and a half-integral threshold offset $m - \frac{1}{2}$. For a genuine two-cycle one has $m \geq 2$; the specialization $m = 1$ loses one core edge and belongs to the acyclic class. These templates therefore give rational but not integer depth-six witnesses.

For the three-component self-loop cases, the feeder form has 32 observable-rank-three row assignments, reducing to six lower maps up to interchange of the isolated states. The width-one through width-three threshold matrix has determinant ± 1 , and exact substitution gives $\Delta_4 = \Delta_5 = \Delta_6 = 0$. The disconnected-edge form has 168 symbolic templates, all satisfying

$$\Delta_6 \in \langle \Delta_1, \dots, \Delta_5 \rangle.$$

These computations are finite certificates for the structural cases used in Theorem 6.3. They are distinct from heuristic local search: row lengths remain symbolic where unbounded, threshold variables are eliminated exactly, and exceptional parameter loci are retained rather than discarded by generic division.

13.5 Reproducibility principle

Because all automata, census matrices, lower maps and affine recurrences are finite and integral, every verification in this section can be performed in exact arithmetic. Numerical spectral calculations are unnecessary for the main complexity theorems. When numerical pole locations are used to explore positivity, they should be regarded only as exploratory unless accompanied by exact root isolation or an independent recurrence proof.

14. Discussion and open problems

The main structural picture is now simple. The residual DFA contributes s census coordinates, the lower-itinerary quotient contributes b translation coordinates, and one common-translation mode is invisible to positional value. This

gives the sharp pair

$$R_U(s, b) = s + b - 1, \quad R_\beta(s, b) = s + b.$$

The remaining questions concern exact recognition radii, arithmetic defects, positivity, and extensions beyond the synchronous setting.

14.1 Exact finite-recognition radii

Theorem 6.1 gives

$$H_K(s, b) \leq s + b - 2, \quad K \in \{\mathbb{Z}, \mathbb{Q}\}.$$

Theorem 6.3 shows that the integer bound is not universally sharp:

$$H_{\mathbb{Z}}(4, 4) = 5 < 6,$$

while Proposition 6.4 gives

$$H_{\mathbb{Q}}(4, 4) = 6.$$

Thus the recognition problem has two layers. The quotient realization controls the universal linear-algebraic horizon, while the local threshold equations may impose arithmetic lattice obstructions that remove some extremal discrepancy modes.

The natural problems are now to determine

$$H_{\mathbb{Z}}(s, b), \quad H_{\mathbb{Q}}(s, b),$$

and to characterize the cells where

$$H_{\mathbb{Z}}(s, b) < H_{\mathbb{Q}}(s, b).$$

Even over $\mathbb{Q}$, an adjacency discrepancy is not an arbitrary output functional of the quotient realization; its census-and-translation form may impose further structural restrictions. The exact recognition-radius problem is therefore distinct from the sharp recurrence-complexity problem solved in Section 7.

14.2 Positive genealogical extremality

The algebraic complexity bounds are sharp without sign restrictions. Positivity is harder. In one-branch genealogical families, extra denominator factors can introduce complex poles inside the census Perron circle, producing sign changes only after long positive prefixes. Finite coefficient checks are therefore not a positivity proof.

The exact maximal Hankel rank of a positive genealogical basis remains open in the unresolved even-state cases.

14.3 Arbitrary regular languages

Prefix closure and right-prolongability make every live residual synchronous in width. Without them, the support sets

$$S_q = \{n \geq 0 : N_q(n) > 0\}$$

can differ between states. A general extension will therefore need to combine residual state, admissible-width phase and affine translation data. This is the natural route toward a broader positionality theory for abstract numeration systems.

14.4 Higher-dimensional positional evaluations

The present theory uses one scalar basis U . Vector-valued or multi-basis evaluations may exist when scalar positionality fails. The residual-affine viewpoint suggests that finite reachable-observable structure should persist, but interval rigidity and uniqueness would need new formulations.

14.5 Further canonical reductions

The lower-itinerary quotient is canonical but need not be the final minimal realization. Reachability and observability may reduce the dimension further for a particular automaton. A useful next problem is to characterize combinatorially when

$$r_U = s + b - 1$$

and when additional cancellations are forced.

14.6 Other canonical root boundaries

Fixed-width and genealogical rank correspond to the root boundaries

$$0 \quad \text{and} \quad B_n = \sum_{m < n} C_m.$$

Other natural graded orders may select other intrinsic boundaries. Classifying such boundaries could reveal further canonical specializations of the same local theory.

15. Conclusion

Normalized scalar positionality in an ordered regular language is controlled by a rigid finite affine structure. Root positionality propagates to every live residual. Width one forces interval digit sets, lower edges determine residual translations, and adjacent digit blocks tile lexicographic rank exactly. Once any branching residual exists, one adjacency determines the unique normalized place-value basis, and the entire residual translation system is integral and intrinsic.

The lower-threshold dynamics admit a canonical finite quotient. If the minimal residual DFA has s states and the lower dynamics have b distinct threshold itineraries, the affine realization compresses to $s + b$ coordinates. The common-translation gauge is invisible to positional values, yielding

$$r_U \leq s + b - 1,$$

while residual boundaries may retain all $s + b$ modes. Both bounds are sharp for every admissible pair:

$$R_U(s, b) = s + b - 1, \quad R_\beta(s, b) = s + b.$$

Fixed-width and genealogical positionality are not separate local theories. They are root-specific boundary conditions on the same intrinsic affine system. Fixed-width rank selects a zero root boundary; genealogical rank selects the cumulative census boundary. For positive bases these two canonical root conditions force Perron-scale growth, whereas a general intrinsic boundary may support strictly faster positional growth.

The resulting picture separates three finite-state phenomena that state count alone obscures: residual census dynamics, lower-threshold translation dynamics, and the global root boundary. This separation gives both a finite recognition theory and an exact two-parameter recurrence-complexity classification. The recognition horizon has an additional arithmetic layer: the ambient bound is attained by the rational $(4, 4)$ relaxation but not by integer digit systems, with $H_{\mathbb{Q}}(4, 4) = 6$ and $H_{\mathbb{Z}}(4, 4) = 5$. Thus reachable-observable dimension alone does not determine the exact finite recognition radius. Extending the same structure beyond synchronous continuation widths to arbitrary regular languages is the principal remaining step toward a general positionality theory for abstract numeration systems.

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