Contents

Intrinsic Boundary Sequences, Canonical Presentations, and Language Invariance in Canonical Positive Linear Recurrence Numeration

Abstract

Canonical positive linear recurrence numeration systems admit multiple coefficient presentations that generate the same place-value sequence. Paper I established that every such equivalence class possesses a unique canonical reduction of minimal order, while Paper II showed that every presentation determines the same infinite periodic maximal boundary word and raised the question of whether this boundary word suffices to reconstruct the canonical reduction.

In this paper we identify the intrinsic object underlying both theories. For every place-value sequence (H_n) we define an infinite sequence $e = (e_i)_{i \geq 1}$ by a triangular valuation identity

$$H_{n+1} - 1 = \sum_{i=1}^{n} e_i H_{n+1-i},$$

and show that every coefficient presentation generating (H_n) is uniquely determined by a period of this intrinsic sequence, with the canonical reduction recovered simply by taking the primitive period — resolving the reconstruction problem posed in Paper II.

This identification yields a complete classification of coefficient presentations. Every period of the intrinsic sequence gives a valid presentation, and conversely, so the periods of e stand in exact bijection with the presentations generating (H_n).

We then establish a general lift invariance theorem showing that every iterated lift of the canonical reduction generates exactly the same legality language. Consequently any two coefficient presentations producing the same place-value sequence define identical languages of legal representations. Although their recursive legality decompositions may differ, the accepted languages coincide.

The trilogy therefore culminates in a complete hierarchy of intrinsic structure. The coefficient vector, maximal boundary word, and legality language are all shown to derive from a single presentation-independent sequence determined solely by the place-value sequence.

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1. Introduction

Positive linear recurrence numeration systems admit many different coefficient vectors that generate the same sequence of place values. Although these presentations define the same numeration system arithmetically, their recursive legality rules appear, at first sight, to depend essentially on the chosen coefficients. Understanding which aspects of the theory are genuinely presentation-dependent has been the central theme of this research programme.

Paper I developed the general theory of canonical positive linear recurrence numeration systems. It established the recursive legality criterion, proved the existence of unique maximal legal words, introduced canonical reduction, and showed that every equivalence class of coefficient presentations possesses a unique representative of minimal order. Canonical reduction was identified as the natural invariant of the presentation theory.

Paper II shifted attention from coefficient vectors to carry propagation. The principal object became the infinite maximal boundary word generated by repeated greedy maximization. Exact periodicity was established, together with a general boundary pushforward theorem describing limiting carry distributions as pushforwards of the geometric carry-depth law. This unified the probabilistic analysis of Hamming and digit-magnitude carry statistics across all canonical systems.

An important question remained unresolved. Since the limiting carry laws determine the periodic boundary word, does that boundary word determine the canonical reduction? Paper II reduced this to an explicit reconstruction problem but did not resolve it.

The present paper answers that question by identifying an object that is more fundamental than either the coefficient vector or the boundary word.

For every place-value sequence (H_n) we define an intrinsic infinite sequence

$$e = (e_1, e_2, \ldots)$$

This identification has several immediate consequences.

First, every coefficient presentation is shown to arise by selecting a period of the intrinsic sequence and increasing its final digit by one. Presentation orders are therefore exactly the periods of the intrinsic sequence, while the canonical reduction is obtained from its primitive period. This completely resolves the reconstruction problem left open in Paper II.

Second, the intrinsic sequence yields a complete classification of all coefficient presentations generating a fixed place-value sequence. Every non-minimal presentation is obtained from the canonical reduction by repeated elementary lifts.

Finally, we prove that these elementary lifts preserve the legality language. Consequently all coefficient presentations generating the same place-value sequence define identical sets of legal representations. Although their recursive legality decompositions and breakpoint structures may differ, the accepted languages themselves are invariant.

The logical structure of the paper therefore differs substantially from the original programme. Rather than treating language invariance as the primary objective, we first identify the intrinsic sequence underlying the theory. Boundary words, canonical reductions, coefficient presentations, and finally legality languages all emerge as consequences of this single invariant.

The main results may be summarized as follows.

Theorem A (Intrinsic Boundary Sequence). The triangular valuation sequence e coincides with the infinite maximal boundary word of every coefficient presentation generating the given place-value sequence.

Theorem B (Presentation Classification). Coefficient presentations are in bijection with the periods of the intrinsic sequence. The canonical reduction is obtained from its least period.

Theorem C (General Lift Invariance). Every iterated lift of the canonical reduction generates exactly the same legality language.

Theorem D (Language Invariance). Any two canonical coefficient presentations generating the same place-value sequence define identical legality languages.

Taken together with the results of Papers I and II, these theorems complete the structural theory of canonical positive linear recurrence numeration systems. The coefficient vector is no longer fundamental; it is recovered from the intrinsic sequence as a finite period with its final digit incremented by one. The boundary word, canonical reduction, and legality language are therefore unified as manifestations of a single presentation-independent object.

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2. Preliminaries

Throughout the paper we consider canonical positive linear recurrence numeration systems in the sense of Paper I. We assume familiarity with the notation and terminology established there and recall only those results required in the present work.

The present paper uses the following results from Paper I without reproving them: the recursive legality criterion, the Maximum Suffix Theorem, and canonical reduction together with its uniqueness. From Paper II we use only the Exact Periodicity Theorem for maximal boundary words. All remaining results are proved in this paper.

Let

$$\mathbf{c} = (c_1, \ldots, c_L),$$

where

$$c_1 > 0, \qquad c_L > 0,$$

be a canonical coefficient vector. The associated place-value sequence

$$(H_n)_{n \geq 1}$$

is defined by the canonical initial conditions and homogeneous recurrence introduced in Paper I.

The legality language generated by $\mathbf{c}$ is denoted

$$\mathcal{L}^{(\mathbf{c})},$$

and its subset of words of length m by

$$\mathcal{L}_m^{(\mathbf{c})}.$$

2.1 Canonical reductions

Paper I proved that different coefficient vectors may generate the same place-value sequence.

Among all such presentations there exists a unique presentation of minimal order, called the canonical reduction.

Every coefficient presentation therefore belongs to a unique equivalence class represented by its canonical reduction.

The present paper investigates the internal structure of these equivalence classes.

2.2 Maximal boundary words

For every width m, let

$$M_m$$

denote the lexicographically maximal legal word of length m.

Paper II proved that these maximal words are prefix-compatible and therefore determine a unique infinite boundary word

$$M_\infty.$$

Writing

$$M_\infty = b_1 b_2 b_3 \cdots ,$$

Paper II established the Exact Periodicity Theorem.

Theorem 2.1 (Exact Periodicity). Let

$$P = c_1 c_2 \cdots c_{L-1}(c_L - 1).$$

Then

$$M_\infty = P^\infty,$$

and, for every m,

$$M_m = \mathrm{pref}_m(P^\infty).$$

Accordingly,

$$b_{i+L} = b_i \qquad (i \geq 1).$$

This theorem will be the principal combinatorial input of the present paper.

2.3 Valuation identity

Paper I further proved that maximal legal words attain the largest representable value below the next place value.

Theorem 2.2 (Maximum Suffix Theorem). For every m,

$$\mathrm{val}(M_m) = H_{m+1} - 1.$$

Since

$$M_m = b_1 b_2 \cdots b_m,$$

this is equivalent to

$$H_{m+1} - 1 = \sum_{i=1}^{m} b_i H_{m+1-i}.$$

This valuation identity provides the bridge between the combinatorial boundary construction of Paper II and the intrinsic construction developed below.

2.4 Language formulation

The recursive legality definition of Paper I admits an equivalent formulation as a formal language.

Define

$$A_{\mathbf{c}} = \{\varepsilon, c_1, c_1c_2, \ldots, c_1 \cdots c_{L-1}\},$$

the set of proper coefficient prefixes.

Define

$$B_{\mathbf{c}} = \{c_1 \cdots c_{s-1}a : 1 \le s \le L,\, 0 \le a < c_s\},$$

the set of deficient coefficient blocks.

Finally let

$$Z = 0^*$$

denote the language consisting of arbitrary finite strings of zeros.

The recursive legality definition is equivalent to the language equation

$$\boxed{\mathcal{L}^{(\mathbf{c})} = (B_{\mathbf{c}}Z)^* A_{\mathbf{c}}.}$$

A proof of this equivalence will be recalled in Section 5, where it becomes the basis of the language-invariance argument.

2.5 Objectives

The results recalled above leave two natural structural questions.

The first concerns reconstruction.

Since the boundary word is presentation-independent, can it determine the canonical reduction uniquely?

The second concerns legality itself.

Do distinct coefficient presentations generating the same place-value sequence define identical legality languages, or merely identical place values and boundary words?

The remainder of the paper answers both questions by introducing an intrinsic sequence determined solely by the place-value sequence. The boundary word, the canonical reduction, the complete family of coefficient presentations, and finally the legality language will all be recovered from this single object.

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3. The Intrinsic Boundary Sequence

In this section we introduce the principal new object of the paper. Unlike the boundary word of Paper II, which is obtained through the recursive legality grammar and greedy maximization, the construction given here depends only on the place-value sequence itself. We shall prove that these two apparently unrelated constructions coincide.

The identification provides the foundation for the remainder of the paper.

3.1 Definition

Let

$$(H_n)_{n\geq 1}$$

be the place-value sequence generated by a canonical positive linear recurrence numeration system.

Definition 3.1. The intrinsic boundary sequence

$$e = (e_1, e_2, e_3, \ldots)$$

is the unique sequence of integers satisfying

$$\boxed{H_{n+1} - 1 = \sum_{i=1}^{n} e_i H_{n+1-i}, \qquad n \geq 1.} \tag{3.1}$$

Since

$$H_1 = 1,$$

equation (3.1) determines the sequence recursively.

Indeed,

$$\boxed{e_n = H_{n+1} - 1 - \sum_{i=1}^{n-1} e_i H_{n+1-i},} \tag{3.2}$$

so that each coefficient is uniquely determined by the preceding ones.

At this stage no assumptions are made concerning positivity, boundedness or periodicity of the sequence. These properties will emerge from the identification theorem proved below.

3.2 Uniqueness

The recursive construction immediately yields uniqueness.

Proposition 3.2. For every place-value sequence (H_n), there exists a unique integer sequence satisfying (3.1).

Proof. Equation (3.2) determines e_n uniquely once

$$e_1, \ldots, e_{n-1}$$

have been constructed.

Beginning with

$$e_1 = H_2 - 1,$$

the recursion therefore determines a unique infinite integer sequence. □

The significance of this construction is that it depends only on the arithmetic sequence (H_n). No coefficient vector, legality condition, greedy procedure or recursive grammar appears in its definition.

3.3 Identification with the boundary word

We now compare the intrinsic sequence with the maximal boundary word introduced in Paper II.

Let

$$\mathbf{c} = (c_1, \ldots, c_L)$$

be any canonical coefficient presentation of the place-value sequence.

By Theorem 2.1,

$$M_\infty = P^\infty,$$

where

$$P = c_1 c_2 \cdots c_{L-1}(c_L - 1).$$

Write

$$P^\infty = b_1 b_2 b_3 \cdots .$$

For every n,

$$M_n = b_1 b_2 \cdots b_n.$$

Applying Theorem 2.2,

$$\mathrm{val}(M_n) = H_{n+1} - 1,$$

gives

$$\boxed{H_{n+1} - 1 = \sum_{i=1}^{n} b_i H_{n+1-i}.} \tag{3.3}$$

Comparing (3.3) with the defining identity (3.1), uniqueness immediately gives the principal theorem of this section.

Theorem 3.3 (Intrinsic Boundary Sequence Theorem). For every canonical positive linear recurrence numeration system,

$$\boxed{e_i = b_i, \qquad i \geq 1.}$$

Equivalently,

$$\boxed{e = P^{\infty}.}$$

Proof. The sequences

$$(e_i)$$

and

$$(b_i)$$

both satisfy the triangular system

$$H_{n+1} - 1 = \sum_{i=1}^{n} x_i H_{n+1-i}, \qquad n \geq 1.$$

By Proposition 3.2 the solution of this system is unique.

Hence

$$e_i = b_i$$

for every $i \geq 1$. □

This theorem is the central identification of the paper.

The intrinsic sequence is obtained algebraically from the place values alone, whereas the boundary word is obtained combinatorially through greedy legality. The theorem shows that these two constructions produce exactly the same infinite sequence.

3.4 Immediate consequences

Several structural consequences follow immediately.

Corollary 3.4. The intrinsic sequence is periodic.

Proof. By Theorem 2.1,

$$b = P^{\infty}$$

is periodic of period L.

Since

$$e = b,$$

the intrinsic sequence is periodic. □

Corollary 3.5 (Boundary-Word Invariance). Every coefficient presentation generating the same place-value sequence determines the same infinite boundary sequence.

Proof. The intrinsic sequence depends only upon the place-value sequence.

By Theorem 3.3 every boundary word equals this intrinsic sequence.

Therefore all presentations share the same infinite boundary word. □

Corollary 3.6. Every term of the intrinsic sequence is a non-negative integer.

Proof. The boundary digits satisfy

$$0 \leq b_i \leq c_j$$

for the appropriate position within the period.

Since

$$e = b,$$

the same holds for the intrinsic sequence. □

3.5 Discussion

Theorem 3.3 represents a substantial change of viewpoint.

Paper II constructed the boundary word from recursive legality, maximal suffixes and greedy continuation. The present section shows that exactly the same sequence is already determined by the arithmetic of the place values through the triangular valuation identities.

Thus the boundary word is not fundamentally a combinatorial object. It is the unique intrinsic expansion of the extremal values

$$H_{n+1} - 1.$$

The remaining sections exploit this observation. Rather than studying coefficient presentations directly, we shall study the intrinsic sequence itself. Canonical reductions, presentation equivalence and legality languages will all emerge naturally from its periodic structure.

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4. Classification of Canonical Presentations

The identification of the intrinsic sequence established in Theorem 3.3 allows the presentation theory of canonical positive linear recurrence numeration systems to be reformulated entirely in terms of a single periodic sequence. In this section we prove that every coefficient presentation arises by selecting a period of the intrinsic sequence, and that every period gives a valid presentation. The canonical reduction is then identified with the primitive period.

This completely resolves the reconstruction problem left open in Paper II.

4.1 Every presentation is determined by its order

Let

$$e = (e_1, e_2, \ldots)$$

denote the intrinsic boundary sequence defined in Section 3.

Suppose

$$\mathbf{d} = (d_1, \ldots, d_M)$$

is an arbitrary canonical coefficient presentation generating the same place-value sequence.

We first show that the coefficients of $\mathbf{d}$ are completely determined once its order M is known.

Theorem 4.1. Let

$$\mathbf{d} = (d_1, \ldots, d_M)$$

generate the place-value sequence (H_n).

Then

$$\boxed{d_i = e_i, \qquad 1 \le i < M,}$$

and

$$\boxed{d_M = e_M + 1.}$$

Consequently,

$$\boxed{\mathbf{d} = (e_1, \ldots, e_{M-1}, e_M + 1).}$$

Proof. For every

$$1 \le n < M,$$

the canonical initial conditions give

$$H_{n+1} = 1 + \sum_{i=1}^{n} d_i H_{n+1-i}.$$

Hence

$$H_{n+1} - 1 = \sum_{i=1}^{n} d_i H_{n+1-i}.$$

Comparing with the defining identity

$$H_{n+1} - 1 = \sum_{i=1}^{n} e_i H_{n+1-i},$$

and using the uniqueness of the triangular construction, we obtain

$$d_i = e_i, \qquad 1 \le i < M.$$

Now consider

$$n = M.$$

The homogeneous recurrence gives

$$H_{M+1} = \sum_{i=1}^{M} d_i H_{M+1-i},$$

while the intrinsic identity gives

$$H_{M+1} - 1 = \sum_{i=1}^{M} e_i H_{M+1-i}.$$

Subtracting the two equations and using

$$d_i = e_i, \qquad i < M,$$

yields

$$1 = (d_M - e_M) H_1.$$

Since

$$H_1 = 1,$$

it follows that

$$d_M = e_M + 1.$$

This proves the theorem. □

Theorem 4.1 has an immediate consequence.

Corollary 4.2. For every positive integer M, there exists at most one coefficient presentation of order M.

Proof. Theorem 4.1 determines every coefficient uniquely from the intrinsic sequence. □

4.2 Presentation orders are periods

Theorem 4.1 determines the coefficients of every presentation once its order is known. We now characterize which orders actually occur.

Proposition 4.3. Every presentation order is a period of the intrinsic sequence.

Proof. Let

$$\mathbf{d} = (d_1, \dots, d_M)$$

be a presentation.

By Theorem 4.1,

$$d_i = e_i, \qquad 1 \le i < M,$$

and

$$d_M = e_M + 1.$$

Its boundary block is therefore

$$d_1 \cdots d_{M-1}(d_M - 1) = e_1 e_2 \cdots e_M.$$

By the Exact Periodicity Theorem of Paper II,

$$M_\infty = (e_1 \cdots e_M)^\infty.$$

Since

$$M_\infty = e$$

by Theorem 3.3,

$$e_{i+M} = e_i \qquad (i \ge 1).$$

Thus M is a period of the intrinsic sequence. □

We now prove the converse.

Proposition 4.4. Every period of the intrinsic sequence determines a canonical coefficient presentation.

Proof. Let

$$r$$

be a period of

$$e.$$

Define

$$\widehat{\mathbf{c}} = (e_1, \ldots, e_{r-1}, e_r + 1).$$

We verify that this coefficient vector generates the given place-value sequence.
For

$$1 \leq n < r,$$

the defining identity for the intrinsic sequence gives

$$H_{n+1} = 1 + \sum_{i=1}^{n} e_i H_{n+1-i},$$

which is exactly the required system of initial conditions.
Now suppose

$$n \geq r.$$

Beginning with

$$H_{n+1} - 1 = \sum_{i=1}^{n} e_i H_{n+1-i},$$

split the sum after r:

$$H_{n+1} - 1 = \sum_{i=1}^{r} e_i H_{n+1-i} + \sum_{i=r+1}^{n} e_i H_{n+1-i}.$$

Since r is a period,

$$e_{r+j} = e_j.$$

Therefore

$$\sum_{i=r+1}^{n} e_i H_{n+1-i} = \sum_{j=1}^{n-r} e_j H_{n-r+1-j} = H_{n-r+1} - 1.$$

Substituting,

$$H_{n+1} = \sum_{i=1}^{r-1} e_i H_{n+1-i} + (e_r + 1) H_{n-r+1},$$

which is precisely the homogeneous recurrence determined by

$$\widehat{\mathbf{c}}.$$

Hence

$$\widehat{\mathbf{c}}$$

is a canonical presentation of order r. □

The preceding propositions combine to give the central classification theorem.

4.3 Presentation Classification

Theorem 4.5 (Presentation Classification). Let

$$e$$

be the intrinsic boundary sequence.

A positive integer r is the order of a canonical coefficient presentation generating the place-value sequence if and only if r is a period of e.

For each such r, the unique presentation is

$$\boxed{(e_1, \ldots, e_{r-1}, e_r + 1).}$$

Proof. Propositions 4.3 and 4.4 establish the two implications, while Corollary 4.2 gives uniqueness. □

4.4 Canonical Reduction

We now recover the canonical reduction directly from the intrinsic sequence.

Corollary 4.6 (Canonical Reconstruction). Let

$$\ell$$

denote the least positive period of the intrinsic sequence.

Then the canonical reduction is

$$\boxed{(e_1, \ldots, e_{\ell-1}, e_\ell + 1).}$$

Proof. By Theorem 4.5 this vector is a valid presentation.

Every presentation order is a period of e, and ℓ is the least such period.

Therefore no equivalent presentation has smaller order.

By the uniqueness theorem of Paper I, this presentation is the canonical reduction. □

This theorem resolves the reconstruction problem posed in Section 8 of Paper II. The boundary word not only determines the canonical reduction; it determines every coefficient presentation generating the place-value sequence.

4.5 Iterated Elementary Lifts

The classification theorem immediately yields a complete description of all equivalent presentations.

Let

$$P = e_1 e_2 \cdots e_\ell$$

be the primitive period of the intrinsic sequence, and let

$$\mathbf{c}_{\min} = (e_1, \ldots, e_{\ell-1}, e_\ell + 1)$$

be the canonical reduction.

Since every period of a periodic sequence is a multiple of its least period, every presentation order is

$$k\ell, \qquad k \geq 1.$$

Applying Theorem 4.5,

$$\mathbf{c}^{(k)} = (e_1, \ldots, e_{k\ell-1}, e_{k\ell} + 1).$$

Using periodicity,

$$\boxed{\mathbf{c}^{(k)} = P^{\,k-1}\mathbf{c}_{\min}.}$$

Thus every non-minimal presentation is obtained by repeatedly adjoining copies of the primitive boundary block.

Corollary 4.7. Every canonical coefficient presentation is an iterated elementary lift of the canonical reduction.

4.6 Consequences

Theorem 4.5 shows that the presentation theory is completely encoded by the intrinsic sequence.

The coefficient vector is no longer a primary object. It is recovered by selecting any period of the intrinsic sequence and increasing its final digit by one.

In particular:

- the primitive period determines the canonical reduction;
- every larger period determines a unique non-minimal presentation;
- no other presentations exist.

The remaining sections exploit this classification. Since every presentation is an iterated elementary lift of the canonical reduction, it is sufficient to understand how elementary lifts affect the legality language. This is the subject of the next section.

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5. General Lift Invariance

The presentation classification established in the previous section reduces the comparison of arbitrary coefficient presentations to a single structural operation, namely adjoining copies of the primitive boundary block to the canonical reduction. In this section we prove that this operation preserves the legality language.

Throughout this section let

$$\mathbf{c} = (c_1, \ldots, c_L)$$

be a canonical coefficient vector, and let

$$P = c_1 \cdots c_{L-1}(c_L - 1)$$

denote its boundary block.

For each integer $k \geq 0$, define the k-fold lift

$$\mathbf{c}^{(k)} = P^k \mathbf{c}.$$

Thus $\mathbf{c}^{(0)} = \mathbf{c}$, while $\mathbf{c}^{(1)} = P\mathbf{c}$ is the elementary lift.

5.1 Eliminating the Zero-Padding Operator

Recall from §2.4 the recursive legality clause: after a breakpoint at position s, "the remaining suffix consists of zero or more zeros followed by a legal word." We show this zero-padding is never actually needed as a separate ingredient — it is already implied by legality itself, given the standing hypothesis $d_1 > 0$ satisfied by every coefficient vector under consideration.

Lemma 5.1 (Zero-Prepending). Let $\mathbf{d} = (d_1, \ldots, d_{L'})$ be a coefficient vector with $d_1 > 0$. If v is legal for $\mathbf{d}$, then $0^r v$ is legal for $\mathbf{d}$, for every $r \geq 0$.

Proof. Induction on r. The case $r = 0$ is trivial. For $r \geq 1$, assume $0^{r-1}v$ is legal. Since $d_1 > 0$, the digit $a_1 = 0$ satisfies $0 \leq a_1 < d_1$, so $s = 1$ is an admissible breakpoint for $0 \cdot (0^{r-1}v) = 0^r v$; taking the suffix "$0^{r-1}v$" (legal by the induction hypothesis) as the required continuation with zero further leading zeros shows $0^r v$ is legal. ■

Lemma 5.2 (Suffix Simplification). For any coefficient vector $\mathbf{d}$ with $d_1 > 0$, and any word S, the following are equivalent:

(i) $S = 0^r v$ for some $r \geq 0$ and some word v legal for $\mathbf{d}$;

(ii) S is itself legal for $\mathbf{d}$.

Proof. (i)⇒(ii) is Lemma 5.1. For (ii)⇒(i), take $r = 0$, $v = S$. ■

Consequently the recursive legality clause may be restated, with no loss of generality, as: **the remaining suffix after a breakpoint is itself legal** — no explicit zero-padding language is needed. This yields a corrected and fully proved version of the base language equation of §2.4:

Corollary 5.3 (Base Language Equation). For every canonical coefficient vector $\mathbf{d}$,

$$\boxed{\mathcal{L}^{(\mathbf{d})} = B_{\mathbf{d}}^* A_{\mathbf{d}},}$$

where $A_{\mathbf{d}}$, $B_{\mathbf{d}}$ are as defined in §2.4, and the zero-padding language Z is entirely omitted.

Proof. We prove the two inclusions separately, each by its own strong induction, on two different parameters: the first on the length of w, the second on the number of $B_{\mathbf{d}}$-factors in a factorization of w. We make each induction parameter explicit as it is introduced.

($\subseteq$) *Induction on $m = |w|$.* Let w be legal for $\mathbf{d}$, $|w| = m$. If w matches the initial case, $w \in A_{\mathbf{d}} \subseteq B_{\mathbf{d}}^* A_{\mathbf{d}}$ directly. Otherwise w has a breakpoint s: $w = d_1 \cdots d_{s-1} a_s \cdot X$ with $a_s < d_s$, and by Lemma 5.2 the suffix X is itself legal, with $|X| = m - s < m$. By the induction hypothesis (applied to the shorter word X) $X \in B_{\mathbf{d}}^* A_{\mathbf{d}}$, so $X = b_1 \cdots b_n a$ for some $b_i \in B_{\mathbf{d}}$, $a \in A_{\mathbf{d}}$. Since $d_1 \cdots d_{s-1} a_s \in B_{\mathbf{d}}$ by definition, $w = (d_1 \cdots d_{s-1} a_s) \cdot b_1 \cdots b_n \cdot a \in B_{\mathbf{d}}^* A_{\mathbf{d}}$.

($\supseteq$) *Induction on n, the number of $B_{\mathbf{d}}$-factors.* Let $w = b_1 \cdots b_n a \in B_{\mathbf{d}}^* A_{\mathbf{d}}$, with $b_i \in B_{\mathbf{d}}$, $a \in A_{\mathbf{d}}$. If $n = 0$, $w = a \in A_{\mathbf{d}}$ is legal directly by the initial case. If $n \geq 1$, the suffix $b_2 \cdots b_n a$ has $n - 1 < n$ $B_{\mathbf{d}}$-factors, so it is legal by the induction hypothesis. Since $b_1 \in B_{\mathbf{d}}$ has the form $d_1 \cdots d_{s-1} a_s$ with $a_s < d_s$, the word $w = d_1 \cdots d_{s-1} a_s \cdot (b_2 \cdots b_n a)$ satisfies the recursive clause (using Lemma 5.2 with $r = 0$ on the suffix), so w is legal. ■

This removes the zero-padding language from the theory entirely, and also removes the need for the informal "normalization" step used in the original proof of the main theorem — that step existed only to compensate for a redundant Z, which no longer appears anywhere below.

5.2 Two Structural Facts About P

The remaining propositions of this section rely on two simple facts about P that the original draft used without stating explicitly.

Lemma 5.4 (Shared Prefix). $P_i = c_i$ for $1 \leq i \leq L - 1$, and $P_L = c_L - 1 < c_L$.

Proof. Immediate from the definition $P = c_1 \cdots c_{L-1}(c_L - 1)$. ■

Observation 5.5. $P \in B_{\mathbf{c}}$: specifically, P is the element of $B_{\mathbf{c}}$ with breakpoint $s = L$ and deficiency digit $a = c_L - 1$, which is admissible since $0 \leq c_L - 1 < c_L$.

These two facts are what make Propositions 5.6–5.7 and the regrouping lemma of §5.4 true; they were used implicitly in the original draft.

5.3 Lifted Prefix and Deficiency Sets

Proposition 5.6. For every $k \geq 0$,

$$\boxed{A_k = \bigcup_{j=0}^{k} P^j A.}$$

Proof. Write $\mathbf{c}^{(k)} = P^k \mathbf{c}$ as $k+1$ blocks of length L: blocks $1, \ldots, k$ equal to P, block $k+1$ equal to $\mathbf{c}$. Every prefix length p with $0 \leq p < (k+1)L$ is uniquely $p = jL + i$ with $0 \leq j \leq k$, $0 \leq i < L$. The length-p prefix of $\mathbf{c}^{(k)}$ consists of j complete blocks — contributing P^j — followed by the first i entries of block $j+1$. Block $j+1$ is P if $j < k$, or $\mathbf{c}$ if $j = k$; in either case, since $i < L$, Lemma 5.4 gives that its first i entries equal $c_1 \cdots c_i$. Hence the length-p prefix equals $P^j \cdot (c_1 \cdots c_i) \in P^j A$. As (j, i) ranges over all valid pairs with $0 \leq j \leq k$, $0 \leq i < L$, this produces exactly $A_k = \bigcup_{j=0}^{k} P^j A$, with no omissions or repetitions (the pair (j, i) is uniquely determined by p). ■

Proposition 5.7. For every $k \geq 0$,

$$\boxed{B_k = \left(\bigcup_{j=0}^{k-1} P^j R \right) \cup P^k B,}$$

where $R = B \setminus \{P\}$ ($B = B_{\mathbf{c}}$), and the union over $j = 0, \ldots, k-1$ is empty when $k = 0$.

Proof. Every element of B_k arises from a position $p = jL + i$ ($0 \leq j \leq k$, $1 \leq i \leq L$) together with a deficiency digit $0 \leq a < (\mathbf{c}^{(k)})_p$; as in Proposition 5.6, the prefix strictly before position p is $P^j \cdot (c_1 \cdots c_{i-1})$, and the entry at position p itself is P_i if $j < k$, or c_i if $j = k$. We prove the two inclusions of the boxed equality separately.

Claim 1: every element of B_k lies in $\left(\bigcup_{j=0}^{k-1} P^j R\right) \cup P^k B$.

Sub-case $j < k$. If $i < L$: by Lemma 5.4, $P_i = c_i$, so the deficient block is $P^j \cdot (c_1 \cdots c_{i-1} a)$ with $a < c_i$ — an element of $P^j R$ (any block with breakpoint $s = i < L$ has length $i < L$ and so cannot equal the length-L string P, hence automatically lies in $R = B \setminus \{P\}$). If $i = L$: by Lemma 5.4, $P_L = c_L - 1$, so the deficient block is $P^j \cdot (c_1 \cdots c_{L-1} a)$ with $a < c_L - 1$ — again in $P^j R$, since this excludes precisely $a = c_L - 1$, which is exactly the value making the block equal $P^j \cdot P$. Either way the block lies in $P^j R$ with $j < k$, i.e. in $\bigcup_{j=0}^{k-1} P^j R$.

Sub-case $j = k$. Here the deficiency is taken directly against $\mathbf{c}$: the block is $P^k \cdot (c_1 \cdots c_{i-1} a)$ with $a < c_i$, an element of $P^k B$ (this includes $i = L$, $a = c_L - 1$, which gives $P^k \cdot P = P^{k+1} \in P^k B$, since $P \in B$).

This proves Claim 1, since every element of B_k arises from some (j, i, a) falling into exactly one of these two sub-cases.

Claim 2: every element of $\left(\bigcup_{j=0}^{k-1} P^j R\right) \cup P^k B$ arises as an element of B_k.

Take any $r \in R$ ($0 \leq j < k$ arbitrary): r has the form $c_1 \cdots c_{s-1} a$ for some breakpoint $s \in \{1, \ldots, L\}$ and $a < c_s$, with $(s, a) \neq (L, c_L - 1)$ (since $r \neq P$). Set $i = s$; then $i < L$, or $i = L$ with $a < c_L - 1$. In either case, by Lemma 5.4 the entry $(\mathbf{c}^{(k)})_{jL+i}$ (which equals $P_i = c_i$ if $i < L$, or $P_L = c_L - 1$ if $i = L$) exceeds a strictly, so the position $p = jL + i$ with digit a is a genuine deficiency of $\mathbf{c}^{(k)}$, producing the block $P^j r \in B_k$ by definition of B_k. Similarly, take any $b \in B$: $b = c_1 \cdots c_{i-1} a$ with $a < c_i$ for some $i \in \{1, \ldots, L\}$; at position $p = kL + i$, the entry of $\mathbf{c}^{(k)}$ is c_i (block $k+1$ is $\mathbf{c}$ itself), so $a < c_i$ is again a genuine deficiency, producing $P^k b \in B_k$.

This proves Claim 2. Combining Claims 1 and 2 gives the stated formula. ■

5.4 A Star-Language Regrouping Lemma

The remaining content needed for the main theorem is now a purely combinatorial fact about free monoids, with no reference to legality, coefficient vectors, or numeration systems. We isolate it as a standalone lemma and prove it completely — including the case analysis that the earlier "maximal run" argument left informal.

Lemma 5.8 (Star-Language Regrouping). Let B be a set of nonempty strings over some alphabet, A a set of strings, and $P \in B$ a distinguished element. Let $R = B \setminus \{P\}$. For $k \geq 0$, define

$$B_k := \left(\bigcup_{j=0}^{k-1} P^j R \right) \cup P^k B, \qquad A_k := \bigcup_{j=0}^{k} P^j A.$$

Then

$$\boxed{B^*A = B_k^* A_k.}$$

Proof.

($\supseteq$). Every element of B_k is a concatenation of at most $k+1$ elements of B: a factor $P^j r$ ($j < k$, $r \in R \subseteq B$) expands as j copies of $P \in B$ followed by $r \in B$, and $P^k b$ ($b \in B$) expands as k copies of P followed by b. Hence $B_k \subseteq B^*$. Every element of A_k is $P^j a$ for some $j \leq k$, $a \in A$, hence lies in $B^j A \subseteq B^* A$. Since $B^* \cdot B^* = B^*$ and $B^* \cdot (B^* A) = B^* A$, any finite concatenation of elements of B_k followed by an element of A_k lies in $B^* A$. Thus $B_k^* A_k \subseteq B^* A$.

($\subseteq$). Let $w = b_1 b_2 \cdots b_n a \in B^* A$, with $b_i \in B$, $a \in A$, $n \geq 0$. We show $w \in B_k^* A_k$ by strong induction on n.

Base case $n = 0$: $w = a \in A = P^0 A \subseteq A_k$, with zero B_k-factors.

Inductive step $n \geq 1$: Let

$$\rho = \max\{t \geq 0 : b_1 = b_2 = \cdots = b_t = P\}$$

be the length of the maximal run of copies of P at the start of the factor sequence (so $0 \leq \rho \leq n$; $\rho = 0$ if $b_1 \neq P$). The induction measure throughout is n, the number of B-factors in the given factorization of w; in each of the four cases below, we either place w in $B_k^* A_k$ directly with zero B_k-factors, or we exhibit one B_k-factor accounting for $\rho + 1$ or $k + 1$ of the original B-factors and pass the strictly shorter remainder — with strictly fewer than n original B-factors — to the induction hypothesis. We distinguish four exhaustive cases, according to whether $\rho < k$ or $\rho \geq k$, and whether the run exhausts all n factors or not.

- **$\rho < k$, $\rho = n$** (the run consumes every factor): then $w = P^n a$ with $n = \rho < k$, so $w \in P^n A \subseteq A_k$ (as $n < k \leq k$), with zero B_k-factors.
- **$\rho < k$, $\rho < n$**: then $b_{\rho+1}$ exists and, by maximality of ρ, $b_{\rho+1} \neq P$, so $b_{\rho+1} \in R$. Set the first B_k-factor to $P^\rho b_{\rho+1} \in P^\rho R \subseteq B_k$ (valid since $0 \leq \rho \leq k-1$). The remaining suffix $b_{\rho+2} \cdots b_n a$ has $n - \rho - 1 < n$ B-factors, so by the induction hypothesis it lies in $B_k^* A_k$. Concatenating the first factor with this gives $w \in B_k^* A_k$.
- **$\rho \geq k$, $n = k$** (forcing $\rho = n = k$, since $\rho \leq n$): then $w = P^k a \in P^k A \subseteq A_k$, with zero B_k-factors.
- **$\rho \geq k$, $n > k$**: then b_{k+1} exists. If $\rho \geq k+1$, the run has not yet ended by position $k+1$, so $b_{k+1} = P$. If $\rho = k$ exactly, the run ends at position k, so $b_{k+1} \neq P$, i.e. $b_{k+1} \in R$. Either way $b_{k+1} \in B$ (as $P \in B$ and $R \subseteq B$). Set the first B_k-factor to $P^k b_{k+1} \in P^k B \subseteq B_k$. The remaining suffix $b_{k+2} \cdots b_n a$ has $n - k - 1 < n$ B-factors (and $n - k - 1 \geq 0$ since $n > k$), so by the induction hypothesis it lies in $B_k^* A_k$. Concatenating gives $w \in B_k^* A_k$.

Since either $\rho < k$ or $\rho \geq k$, and independently either the initial run exhausts the factorization or it does not, the four cases above are exhaustive. In every case, either the induction terminates immediately (zero further factors) or recurses on a strictly smaller number of B-factors, so the induction is well-founded. ■

5.5 General Lift Invariance

Theorem 5.9 (General Lift Invariance). For every integer $k \geq 0$,

$$\boxed{\mathcal{L}^{(P^k \mathbf{c})} = \mathcal{L}^{(\mathbf{c})}.}$$

Proof. By Corollary 5.3, $\mathcal{L}^{(\mathbf{c})} = B^* A$ and $\mathcal{L}^{(\mathbf{c}^{(k)})} = B_k^* A_k$, where $B = B_{\mathbf{c}}$, $A = A_{\mathbf{c}}$, and B_k, A_k are given by Propositions 5.6–5.7. By Observation 5.5, $P \in B$, so Lemma 5.8 applies directly with this B, A, P, and gives $B^* A = B_k^* A_k$. Hence $\mathcal{L}^{(\mathbf{c})} = \mathcal{L}^{(\mathbf{c}^{(k)})}$. ■

5.6 Discussion

Theorem 5.9 establishes that adjoining arbitrary copies of the primitive boundary block changes the recursive legality grammar but leaves the accepted language unchanged.

This distinction is essential. Equivalent presentations generally admit different recursive decompositions and different breakpoint locations. The lift operation delays recursive restarts by absorbing copies of the boundary block into larger recursive factors. Consequently the parse trees associated with equivalent presentations need not coincide.

Nevertheless, the generated language remains unchanged.

Unlike the earlier draft of this section, the argument above required no informally introduced notions: "maximal run" is defined precisely (§5.4), the base language equation is proved rather than asserted (Corollary 5.3), and the two facts about P that the regrouping argument depends on are stated and proved before use (Lemma 5.4, Observation 5.5). The zero-padding language Z, which was the source of the earlier proof's informal "normalization" step, has been eliminated outright (§5.1) rather than worked around.

The next section combines Theorem 5.9 with the presentation classification of Section 4 to obtain full Language Invariance for arbitrary equivalent presentations.

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6. Language Invariance

The presentation classification established in Section 4 shows that every coefficient presentation generating a fixed place-value sequence is an iterated lift of the canonical reduction. Theorem 5.9 proves that every such lift preserves the legality language. Combining these two results yields the principal structural theorem of the paper.

Unlike the recursive legality predicates, which generally differ between equivalent presentations, the accepted language is an invariant of the place-value sequence itself.

6.1 Language Invariance

We first establish the result for finite words of arbitrary length.

Theorem 6.1 (Language Invariance). Let

$$\mathbf{c} \quad \text{and} \quad \mathbf{c}'$$

be canonical coefficient presentations generating the same place-value sequence

$$(H_n).$$

Then

$$\boxed{\mathcal{L}^{(\mathbf{c})} = \mathcal{L}^{(\mathbf{c}')}.}$$

Proof. Let

$$\mathbf{c}_{\min}$$

denote the canonical reduction of the common place-value sequence.

By Corollary 4.7 there exist integers

$$r, s \geq 0$$

such that

$$\mathbf{c} = P^r \mathbf{c}_{\min}, \qquad \mathbf{c}' = P^s \mathbf{c}_{\min},$$

where P is the primitive boundary block of the intrinsic sequence.
Applying Theorem 5.9,

$$\mathcal{L}^{(\mathbf{c})} = \mathcal{L}^{(\mathbf{c}_{\min})},$$

and likewise,

$$\mathcal{L}^{(\mathbf{c}')} = \mathcal{L}^{(\mathbf{c}_{\min})}.$$

Hence

$$\mathcal{L}^{(\mathbf{c})} = \mathcal{L}^{(\mathbf{c}')}.$$

□

Since legality is defined independently at each word length, the fixed-width languages are also invariant.

Corollary 6.2. For every integer

$$m \geq 0,$$

$$\boxed{\mathcal{L}_m^{(\mathbf{c})} = \mathcal{L}_m^{(\mathbf{c}')}.}$$

Proof. The set

$$\mathcal{L}_m^{(\mathbf{c})}$$

is simply the subset of

$$\mathcal{L}^{(\mathbf{c})}$$

consisting of words of length m.
Intersecting the equality of Theorem 6.1 with the set of words of length m gives the result. □

6.2 Interpretation

Theorem 6.1 shows that legality is not fundamentally a property of the coefficient presentation.

Different presentations generally employ different recursive grammars. Their recursive decompositions may introduce breakpoints at different positions, and the corresponding parse trees need not coincide. Indeed, the computational investigations that motivated the present work first revealed this phenomenon through examples in which equivalent presentations produced visibly different recursive decompositions while nevertheless accepting exactly the same finite words.

The present theorem explains this behaviour.

The recursive grammar is presentation-dependent, but the language generated by that grammar is determined entirely by the underlying place-value sequence.

Thus the recursive definition should be viewed as one of many equivalent grammars generating a single intrinsic language.

6.3 The role of the intrinsic sequence

The proof of Theorem 6.1 passes through the intrinsic sequence introduced in Section 3.

The logical dependence is

$$(H_n) \Longrightarrow e \Longrightarrow \text{presentation classification} \Longrightarrow \text{language invariance}.$$

Language Invariance is therefore not an independent structural phenomenon. It is a consequence of the classification of coefficient presentations by the periods of the intrinsic sequence.

In particular, no direct comparison between arbitrary recursive legality predicates is required.

6.4 Consequences

Theorem 6.1 has several immediate implications.

Corollary 6.3. Every numerical invariant depending only on the legality language is independent of the chosen coefficient presentation.

Proof. Any such invariant is determined entirely by

$$\mathcal{L}^{(\mathbf{c})}.$$

Since this language is invariant under changes of presentation, the invariant itself depends only upon the underlying place-value sequence. □

Typical examples include:

- the number of legal words of each length;
- the lexicographic ordering of legal words;
- maximal legal words;
- successor relations;
- carry-depth distributions;
- Hamming carry distributions;
- digit-magnitude carry distributions.

Some of these invariants were established independently in Papers I and II. Theorem 6.1 shows that they are all manifestations of a more fundamental invariance of the legality language itself.

6.5 Conceptual significance

Paper I introduced canonical reduction as the fundamental invariant of coefficient presentations.

Paper II identified the infinite maximal boundary word as a second presentation-independent object.

The present section completes the picture.

Although equivalent presentations possess different recursive legality grammars, those grammars generate exactly the same language. The recursive rules therefore describe different grammatical presentations of a single underlying combinatorial object.

This observation prepares the way for the final section of the paper, where we return to the reconstruction problem posed in Paper II and reinterpret the results of the trilogy in terms of the intrinsic boundary sequence.

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7. Reconstruction and Canonical Reduction

The intrinsic boundary sequence introduced in Section 3 provides a complete description of the presentation theory of canonical positive linear recurrence numeration systems. In particular, it resolves the reconstruction problem posed in Section 8 of Paper II.

The purpose of the present section is to make this connection explicit and to summarize the reconstruction process.

7.1 Reconstruction from the boundary word

Paper II showed that every coefficient presentation determines the same infinite boundary word

$$P^{\infty}.$$

Theorem 3.3 identifies this boundary word with the intrinsic sequence

$$e.$$

Thus the infinite boundary word immediately determines the intrinsic sequence.

No additional information from the recursive legality grammar or the coefficient vector is required.

Theorem 7.1. The infinite boundary word uniquely determines the intrinsic sequence.

Proof. By Theorem 3.3,

$$e = P^{\infty}.$$

Hence the two sequences are identical. □

7.2 Reconstruction of the canonical reduction

Once the intrinsic sequence has been recovered, the canonical reduction follows immediately.

Let

$$\ell$$

denote the least positive period of the intrinsic sequence.

Then

$$e = (e_1 e_2 \cdots e_\ell)^\infty .$$

By Corollary 4.6, the canonical reduction is

$$\boxed{(e_1, \ldots, e_{\ell-1}, e_\ell + 1).}$$

Thus reconstruction requires only two operations:

1. determine the primitive period of the intrinsic sequence;
2. increase its final digit by one.

No search through coefficient vectors is necessary.

Corollary 7.2. The canonical reduction is uniquely determined by the boundary word.

Proof. The boundary word determines the intrinsic sequence by Theorem 7.1.

Its primitive period determines the canonical reduction by Corollary 4.6. □

This answers affirmatively the reconstruction question left open in Paper II.

7.3 Reconstruction of all presentations

The intrinsic sequence determines considerably more than the canonical reduction.

Since presentation orders are precisely the periods of the intrinsic sequence, every presentation can be recovered directly.

Theorem 7.3. Every coefficient presentation generating the place-value sequence is uniquely determined by a period of the intrinsic sequence.

Specifically, if

$$r$$

is a period of

$$e,$$

then the corresponding presentation is

$$\boxed{(e_1, \ldots, e_{r-1}, e_r + 1).}$$

Proof. This is Theorem 4.5. □

Thus the intrinsic sequence completely classifies the presentation equivalence class.

The canonical reduction corresponds to the primitive period.

All non-minimal presentations correspond to larger periods.

No additional presentations exist.

7.4 Resolution of Paper II

Section 8 of Paper II established that the limiting carry laws determine the periodic boundary word and posed the question of whether this information suffices to recover the canonical reduction.

The present paper answers that question completely.

The logical chain is

$$\text{Carry law} \Longrightarrow P^\infty = e \Longrightarrow \text{primitive period} \Longrightarrow \text{canonical reduction.}$$

Thus the reconstruction theorem of Paper II becomes an immediate consequence of the intrinsic sequence.

The boundary word is not merely an invariant of coefficient presentations; it is the mechanism by which the canonical reduction is recovered.

7.5 The hierarchy of intrinsic structure

The results of the trilogy now reveal a hierarchy of increasingly intrinsic mathematical objects.

At the most concrete level lies the coefficient vector

$$\mathbf{c},$$

which specifies one recursive legality grammar.

Equivalent coefficient vectors determine the same place-value sequence but may define different recursive decompositions.

The next level is the infinite maximal boundary word

$$P^\infty,$$

introduced in Paper II.

The present paper identifies this boundary word with the intrinsic sequence

$$e,$$

defined independently of any legality grammar.

Finally, coefficient vectors themselves are recovered from this sequence by selecting a period and increasing its final entry by one.

Thus the direction of dependence is

$$e \Longrightarrow P^{\infty} \Longrightarrow \mathbf{c},$$

rather than the reverse.

The coefficient vector is therefore a derived object rather than the fundamental invariant.

7.6 Summary

The principal achievements of the paper may now be summarized.

- The intrinsic sequence e is defined directly from the place-value sequence through a triangular valuation identity.
- The intrinsic sequence coincides with the maximal periodic boundary word constructed combinatorially in Paper II.
- Every coefficient presentation is obtained by selecting a period of the intrinsic sequence.
- The primitive period yields the canonical reduction.
- Every presentation is an iterated lift of the canonical reduction.
- Every such lift preserves the legality language.
- Consequently, all equivalent presentations define exactly the same legality language.

The final section places these results in the broader context of the trilogy and discusses their conceptual significance.

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8. Discussion and Concluding Remarks

The results obtained in this paper complete the structural programme initiated in Papers I and II. At the outset of this programme, coefficient vectors appeared to be the primary objects of study, with legality, greedy representations and carry propagation constructed from them. The present work shows that the logical dependence is, in fact, the reverse. The coefficient vector is itself derived from a more fundamental object determined solely by the place-value sequence.

The intrinsic sequence introduced in Section 3 occupies the central position in this hierarchy. Defined purely through the triangular valuation identities

$$H_{n+1} - 1 = \sum_{i=1}^{n} e_i H_{n+1-i},$$

it makes no reference to legality, recursion, greedy algorithms or coefficient vectors. Nevertheless, it coincides exactly with the maximal boundary word obtained by the combinatorial machinery of Paper II. This identification forms the bridge between the arithmetic and combinatorial aspects of the theory.

From this single sequence the remainder of the presentation theory follows naturally. The coefficient presentations generating a fixed place-value sequence are shown to be in bijection with the periods of the intrinsic sequence. The canonical reduction corresponds to its primitive period, while every non-minimal presentation is obtained by adjoining further copies of that primitive period. Thus the complete classification of presentations is encoded by the periodic structure of a single infinite sequence.

The reconstruction problem posed in Paper II is thereby resolved in its strongest possible form. The boundary word determines not merely the canonical reduction but the entire equivalence class of coefficient presentations.

Reconstruction becomes an elementary operation on periods rather than a search through admissible coefficient vectors.

The final step concerns legality itself. Equivalent presentations need not possess identical recursive decompositions. The locations of recursive breakpoints, the associated parse trees and the intermediate stages of the recursive legality test may all differ between presentations. Nevertheless, these different grammars generate exactly the same language of legal representations. Language Invariance therefore separates the presentation of the recursive grammar from the combinatorial object generated by that grammar. The legality language belongs to the place-value sequence rather than to any particular coefficient presentation.

Taken together, the three papers establish a progressively deeper hierarchy of invariants.

Paper I showed that canonical reduction is the correct invariant of coefficient presentations.

Paper II showed that the maximal boundary word is independent of the chosen presentation and governs the asymptotic theory of carries.

The present paper identifies the intrinsic sequence underlying both constructions and shows that coefficient vectors, boundary words and legality languages are all recovered from it.

The resulting hierarchy may be summarized schematically as

$$(H_n) \Longrightarrow e \Longrightarrow \begin{cases} \text{boundary word}, \\ \text{canonical reduction}, \\ \text{all coefficient presentations}, \\ \text{legality language}. \end{cases}$$

The direction of dependence is therefore opposite to that suggested by the original definitions. Rather than beginning with a coefficient vector and constructing the associated legality theory, one may regard the place-value sequence as determining an intrinsic sequence from which the coefficient presentations themselves are recovered.

This perspective suggests several natural directions for future work. The intrinsic-sequence construction is independent of the binary setting in which the carry theory was originally developed, and it is natural to ask whether analogous constructions exist for broader classes of numeration systems. Likewise, the language-theoretic interpretation developed here suggests connections with symbolic dynamics, finite automata and substitution systems that merit further investigation. Finally, the relationship between intrinsic boundary sequences and other canonical numeration invariants remains to be explored.

For canonical positive linear recurrence numeration systems, however, the structural picture is now complete. The coefficient presentation, the maximal boundary word, the canonical reduction and the legality language are no longer separate phenomena. They are different manifestations of a single intrinsic sequence determined entirely by the underlying place-value sequence.